## 14.1 Functions of Several Variables

- Domain & Range
- Interior Point, Boundary Point
- Open Set, Closed Set, Bounded Set
- Level Curves : $\boxed{f(x,y)=c}$
- Level Surfaces : $\boxed{f(x,y,z)=c}$

## 14.2 Limits & Continuity in Higher Dimensions

- Major difference from Calculus I limits (one variable) : a point $(x,y)$ can approach a point $(x_0,y_0)$ in the plane from infinitely many directions, along infinitely many paths. They must all agree in order for the limit to exist!
- Two-Path Test : If $f(x,y)$ has 2 different limits along 2 different paths in the domain as $(x,y)$ approaches $(x_0,y_0)$, then $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ does not exist.

- Typical Examples :

① Continuous functions (aka "plug it in, nothing bad happens") :

$$\lim_{(x,y)\to(0,1)} \frac{x-xy+3}{x^2y+5xy-y^3} = \boxed{-3}$$

② 0/0 limits where we factor & cancel terms

$$\lim_{(x,y)\to(0,0)} \frac{x^2-xy}{\sqrt{x}-\sqrt{y}} = \lim_{(x,y)\to(0,0)} \frac{x(x-y)}{\sqrt{x}-\sqrt{y}} = \lim_{(x,y)\to(0,0)} \frac{x\cancel{(\sqrt{x}-\sqrt{y})}(\sqrt{x}+\sqrt{y})}{\cancel{\sqrt{x}-\sqrt{y}}} = \boxed{0}.$$

③ Two-Path Test (Limit DNE) :

$$f(x,y) = \frac{2x^2y}{x^4+y^2} \text{ has no limit as } (x,y)\to(0,0)$$

(approach along lines) $$f(x,y)\Big|_{y=kx} = \frac{2x^2(kx)}{x^4+(kx)^2} = \frac{2kx^3}{x^4+k^2x^2} = \frac{2kx}{x^2+k^2} \xrightarrow[x\to 0]{} \boxed{0}$$

inconclusive

(approach along parabolas) $$f(x,y)\Big|_{y=kx^2} = \frac{2x^2(kx^2)}{x^4+(kx^2)^2} = \frac{2kx^4}{x^4+k^2x^4} = \frac{2k}{1+k^2} \xrightarrow[x\to 0]{} \boxed{\frac{2k}{1+k^2}}$$

Taking $k=0$ and $k=2$, for example, $\lim_{(x,y)\to(0,0)} f(x,y)$ DNE by the 2-Path Test.

## ④ Switching to Polar Coord. (to show that a limit does exist).

- $$\lim_{(x,y)\to(0,0)} \left( xy \, \frac{x^2-y^2}{x^2+y^2} \right) = \lim_{r\to 0} \left( (r\cos\theta)(r\sin\theta) \frac{(r\cos\theta)^2-(r\sin\theta)^2}{(r\cos\theta)^2+(r\sin\theta)^2} \right)$$
$$= \lim_{r\to 0} r^2 \underbrace{\cos\theta \sin\theta \left(\cos^2\theta - \sin^2\theta\right)}_{\text{bounded}}$$
$$= \boxed{0}.$$

- $$\lim_{(x,y)\to(0,0)} \left[ \ln\left( \frac{7x^2 - x^2y^2 + 7y^2}{x^2+y^2} \right) \right] = \lim_{r\to 0} \left[ \ln \frac{7r^2\cos^2\theta - r^4\cos^2\theta\sin^2\theta + 7\sin^2\theta\, r^2}{r^2\cos^2\theta + r^2\sin^2\theta} \right]$$
$$= \lim_{r\to 0} \left[ \ln \frac{7r^2(\cos^2\theta+\sin^2\theta) - r^4\cos^2\theta\sin^2\theta}{r^2} \right]$$
$$= \lim_{r\to 0} \left[ \ln\left( 7 - \underbrace{r^2 \underbrace{\cos^2\theta\sin^2\theta}_{\text{bounded}}}_{\xrightarrow{r\to 0}\ 0} \right) \right] = \boxed{\ln(7)}$$

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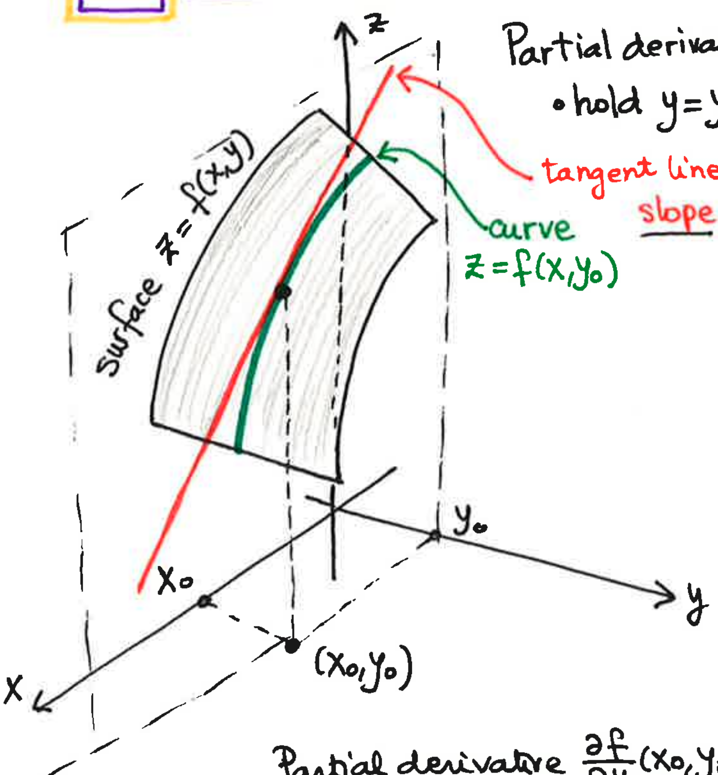
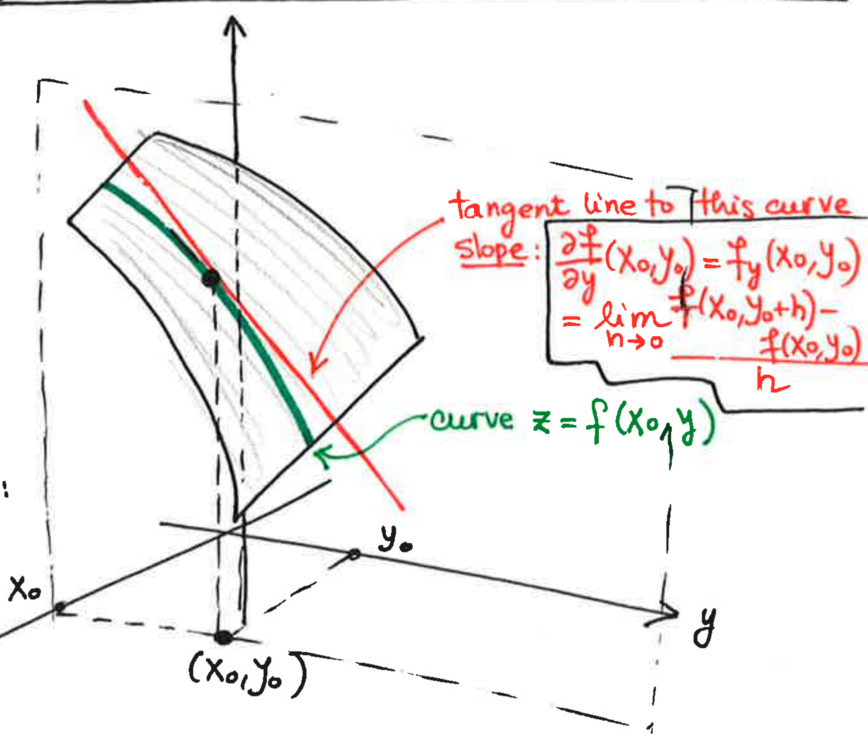
## 14.3 Partial Derivatives

Partial derivative $\frac{\partial f}{\partial x}(x_0,y_0)$:
- hold $y=y_0$ constant




slope: $$\boxed{\frac{\partial f}{\partial x}(x_0,y_0) = f_x(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0+h,y_0)-f(x_0,y_0)}{h}}$$

Partial derivative $\frac{\partial f}{\partial y}(x_0,y_0)$:
- hold $x=x_0$ constant




slope: $$\boxed{\frac{\partial f}{\partial y}(x_0,y_0) = f_y(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0,y_0+h)-f(x_0,y_0)}{h}}$$

- To compute $\frac{\partial f}{\partial x}$ : hold the other variables ($y, z$ etc.) constant & differentiate by $x$.

Ex: $f(x,y,z) = x^2 y + x\sin(yz) + \cos(xyz)$

To find $\frac{\partial f}{\partial x}$, think of $f$ as: $f(x,y,z) = x^2 y + x\sin(yz) + \cos(xyz)$
(green = variable)

$$\frac{\partial f}{\partial x} = 2xy + \sin(yz) + \left(-\sin(xyz)\cdot yz\right)$$

To find $\frac{\partial f}{\partial y}$, think of $f$ as: $f(x,y,z) = x^2 y + x\sin(yz) + \cos(xyz)$

$$\frac{\partial f}{\partial y} = x^2 + x\cos(yz)\cdot z + \left(-\sin(xyz)\cdot xz\right)$$

To find $\frac{\partial f}{\partial z}$, think of $f$ as: $f(x,y,z) = x^2 y + x\sin(yz) + \cos(xyz)$

$$\frac{\partial f}{\partial z} = x\cos(yz)\cdot y + \left(-\sin(xyz)\cdot xy\right)$$

- Higher Order Partial Derivatives :

$$\frac{\partial^2 f}{\partial x^2} = f_{xx} \ ; \ \frac{\partial^2 f}{\partial y^2} = f_{yy} \ ; \ \frac{\partial^2 f}{\partial x \partial y} = f_{yx} \ ; \ \frac{\partial^2 f}{\partial y \partial x} = f_{xy} \ ;$$

$$\frac{\partial^3 f}{\partial x \partial y \partial z} = f_{zyx} \ ; \ \frac{\partial^4 f}{\partial x \partial y^3} = f_{yyyx} \ ; \text{ etc. etc.}$$

- Mixed Derivative Theorem :

If $f(x,y)$ and its partial derivatives $f_x, f_y, f_{xy}, f_{yx}$ are defined throughout an open region containing a point $(x_0, y_0)$ and are all continuous at $(x_0,y_0)$, then $f_{xy}(x_0,y_0) = f_{yx}(x_0,y_0)$.

## 14.4 The Chain Rule :

$w = w(x,y,z)$
$x = x(t)$
$y = y(t)$
$z = z(t)$

$$\boxed{\begin{aligned}\frac{dw}{dt} &= \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} \\ &= w_x \cdot x'(t) + w_y \cdot y'(t) + w_z \cdot z'(t)\end{aligned}}$$

$w = w(x,y,z)$
$x = x(u,v)$, $y = y(u,v)$, $z = z(u,v)$

$$\boxed{\begin{aligned}\frac{\partial w}{\partial u} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u} \\ &= w_x \cdot x_u + w_y \cdot y_u + w_z \cdot z_u\end{aligned}}$$

$$\boxed{\begin{aligned}\frac{\partial w}{\partial v} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v} \\ &= w_x \cdot x_v + w_y \cdot y_v + w_z \cdot z_v\end{aligned}}$$

• **Implicit Differentiation Simplified**: If $F(x,y)=0$ defines $y$ implicitly as a differentiable function of $x$, then $\boxed{\frac{dy}{dx} = -\frac{F_x}{F_y}}$

Ex.: $6x^3-7y^2-xy=0$; find $\frac{dy}{dx}$

Old way:

$$\frac{d}{dx}(6x^3-7y^2-xy) = \frac{d}{dx}(0)$$

$$18x^2 - 14y\left(\frac{dy}{dx}\right) - y - x\left(\frac{dy}{dx}\right) = 0$$

$$18x^2 - y = (x+14y)\frac{dy}{dx}$$

$$\frac{18x^2-y}{x+14y} = \frac{dy}{dx}$$

New way: $F(x,y) = 6x^3-7y^2-xy$

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{18x^2-y}{-14y-x} = \frac{18x^2-y}{14y+x}.$$

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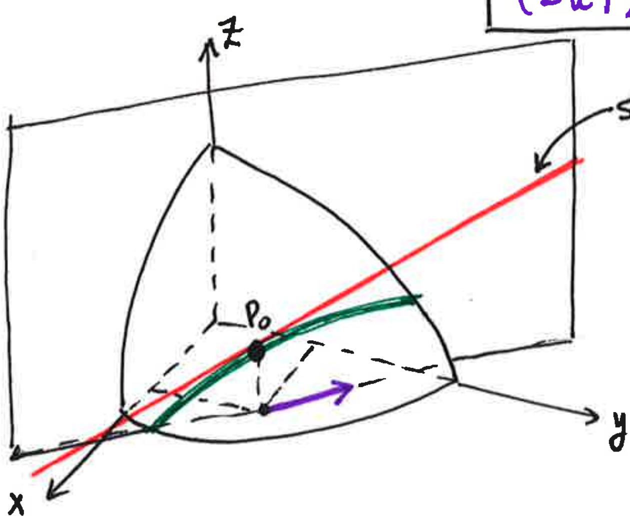
## 14.5 Directional Derivatives & Gradient Vectors

$f(x,y,z)$; Gradient Vector: $\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$

$f(x,y)$; Gradient Vector: $\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j}$

Directional Derivative of $f$ at $P_0$ in the direction of the unit vector $\vec{u}$:

$$\boxed{(D_{\vec{u}}f)_{P_0} = (\nabla f)_{P_0} \cdot \vec{u}}$$




slope $= (D_{\vec{u}}f)_{P_0}$ = rate of change in the direction of $\vec{u}$.

$$D_{\vec{u}}f = |\nabla f||\vec{u}|\cos\theta = |\nabla f|\cos\theta$$

- $f \nearrow$ most rapidly in the direction of $\nabla f$ $(\theta = 0)$; directional deriv. there is $|\nabla f|$.
- $f \searrow$ most rapidly in the direction of $-\nabla f$ $(\theta = \pi)$; directional deriv. there is $-|\nabla f|$.

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## 14.6 Tangent Planes & Normal Lines

Tangent plane to $f(x,y,z)=c$ at $P_0(x_0,y_0,z_0)$: $\boxed{f_x(P_0)(x-x_0) + f_y(P_0)(y-y_0) + f_z(P_0)(z-z_0) = 0}$

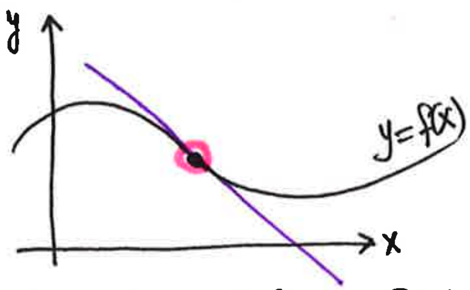
Normal line to $f(x,y,z)=c$ at $P_0(x_0,y_0,z_0)$:

$$\boxed{\begin{aligned} x &= x_0 + f_x(P_0)t \\ y &= y_0 + f_y(P_0)t \\ z &= z_0 + f_z(P_0)t \end{aligned}}$$

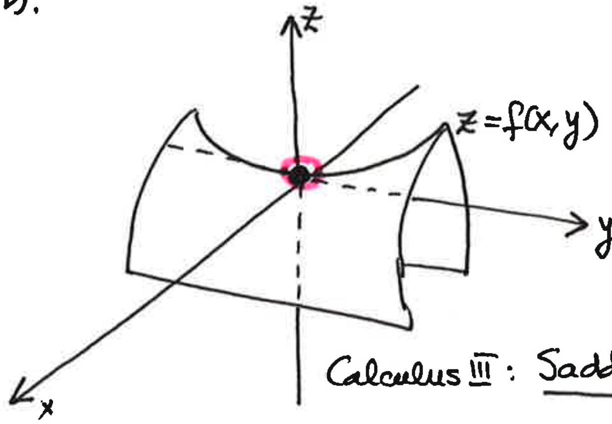
# 14.7 Extreme Values & Saddle Points

**Def.:** A point $(a,b)$ is called a critical point of $f(x,y)$ if $(a,b)$ is an interior point to the domain of $f$ and either $f_x(a,b) = f_y(a,b) = 0$ or one or both $f_x, f_y$ do not exist at $(a,b)$.

**Def.:** A saddle point for $f(x,y)$ is a critical point $(a,b)$ such that in every open disk centered at $(a,b)$ there are domain points $(x,y)$ such that $f(x,y) > f(a,b)$ and points where $f(x,y) < f(a,b)$.




Calculus I & II: Inflection Point




Calculus III: Saddle Point

## First Derivative Test:

If $f(x,y)$ has a local min. or max at an interior point $(a,b)$ and $f_x(a,b)$, $f_y(a,b)$ exist, then $f_x(a,b) = f_y(a,b) = 0$.

In other words: local extrema occur at critical points or at boundary points.

## Second Derivative Test:

Suppose $f_x(a,b) = f_y(a,b) = 0$ at an interior point $(a,b)$. Then $(a,b)$ is a:

$\boxed{(a,b)}$

- local extremum if $\boxed{f_{xx} f_{yy} - f_{xy}^2 > 0}$ at $(a,b)$:
  - local max. if $f_{xx} < 0$ at $(a,b)$
  - local min. if $f_{xx} > 0$ at $(a,b)$
- saddle point if $\boxed{f_{xx} f_{yy} - f_{xy}^2 < 0}$ at $(a,b)$.
- ? if $\boxed{f_{xx} f_{yy} - f_{xy}^2 = 0}$

## Min & Max on Closed Bounded Regions:

1. Find the critical points interior to the region & evaluate $f$ there.
2. Find the boundary points where $f$ has local extrema & evaluate $f$ there.
3. Look through the lists & find the absolute min & max.

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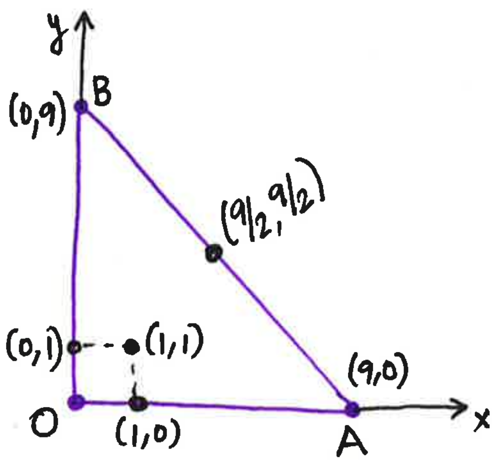
Ex: $f(x,y) = 2+2x+2y-x^2-y^2$; Triangular region in Quad I bounded by $x=0$, $y=0$, $y=9-x$.




**List:**

$\boxed{f(1,1) = 4}$ → Max

$f(1,0) = 3$

$f(0,0) = 2$

$\boxed{f(9,0) = -61}$ → Min

$f(0,1) = 3$

$f(0,9) = -61$

$f(9/2, 9/2) = -\frac{41}{2}$

**1. Interior points:** $f_x = 2-2x$; $f_y = 2-2y$

$$\begin{cases} f_x = 0 \\ f_y = 0 \end{cases} \quad \begin{cases} 2-2x=0 \\ 2-2y=0 \end{cases} \quad \begin{cases} x=1 \\ y=1 \end{cases}$$

Critical point: $(1,1)$

Is it inside the region? Yes. So evaluate $f$ & list it.

**2. Boundary Points:**

(i) $OA$: $y=0$

$f(x,0) = 2+2x-x^2$, $0 \le x \le 9$

Extreme values occur @ boundary pts. or critical points in $(0,9)$

$f'(x,0) = 2-2x$

Critical point: $(1,0)$ – evaluate $f$ here and at the boundary points $(0,0)$, $(9,0)$.

(ii) $OB$: $x=0$

$f(0,y) = 2+2y-y^2$, $0 \le y \le 9$

$f'(0,y) = 2-2y$

Critical point: $(0,1)$; Boundary points: $(0,0)$, $(0,9)$

already done

(iii) $AB$: $y = 9-x$

Already evaluated endpoints $(0,9)$ & $(9,0)$, so we only need to look for critical points.

$$f(x, 9-x) = 2+2x+2(9-x)-x^2-(9-x)^2$$
$$= -61+18x-2x^2$$

$$f'(x, 9-x) = 18-4x$$

$18-4x = 0 \Rightarrow x = 9/2 \Rightarrow y = 9-9/2 = 9/2$

Critical point: $(9/2, 9/2)$

## 14.8 Lagrange Multipliers

Suppose that $f(x,y,z)$ and $g(x,y,z)$ are differentiable and $\nabla g \neq \vec{0}$ when $g(x,y,z)=0$.
To find local min & max values of $f$ subject to the constraint $g(x,y,z)=0$

find the values of $x,y,z$ and $\lambda$ that satisfy:

$$\boxed{\begin{cases} \nabla f = \lambda(\nabla g) \\ g(x,y,z)=0 \end{cases}}$$

Two constraints: To find local extrema of a differentiable $f(x,y,z)$ subject to the constraints $g_1(x,y,z)=0$ and $g_2(x,y,z)=0$ and $g_1, g_2$ are differentiable ($\nabla g_1$ not parallel to $\nabla g_2$), find the values of $x, y, z, \lambda_1, \lambda_2$ that satisfy:

$$\begin{cases} \nabla f = \lambda_1(\nabla g_1) + \lambda_2(\nabla g_2) \\ g_1(x,y,z)=0 \\ g_2(x,y,z)=0 \end{cases}$$

Example: Find the max. value of $f(x,y) = 58 - x^2 - y^2$ on the line $x+7y=50$.

$g(x,y) = x+7y-50$

$\nabla f = \langle -2x, -2y \rangle$

$\nabla g = \langle 1, 7 \rangle$

Solve: $\begin{cases} -2x = \lambda \\ -2y = 7\lambda \\ x+7y-50=0 \end{cases}$ $\quad \begin{cases} x = -\lambda/2 \\ y = -7\lambda/2 \\ x+7y-50=0 \end{cases}$ $\quad -\frac{\lambda}{2} + 7\left(-\frac{7\lambda}{2}\right) - 50 = 0$

$\lambda + 49\lambda + 100 = 0$

$50\lambda = -100$

$\boxed{\lambda = -2}$

$\Rightarrow \boxed{x = +1,\ y = +7}$

The extreme value occurs at $(1,7)$, where:

$f(1,7) = \boxed{8}$.