Name: ______________________________

November 4^{th}, 2015.
Math 2552; Sections F1 – F4; L1 – L4.
Georgia Institute of Technology
Sample Exam 2

I commit to uphold the ideals of honor and integrity by refusing to betray the trust bestowed upon me as a member of the Georgia Tech community. By signing my name below I pledge that I have neither given nor received help on this exam.

Pledged: ______________________________

Problem	Possible Score	Earned Score
1	10	
2	10	
3	20	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
Total	100	

Remember that you must SHOW YOUR WORK to receive credit!

Good luck!

Laplace transforms of some basic functions

$\mathscr{L}\{1\} = \frac{1}{s};\ \ s > 0$

$\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}};\ \ s > 0$

$\mathscr{L}\{e^{kt}\} = \frac{1}{s-k};\ \ s > k$

$\mathscr{L}\{\sin(kt)\} = \frac{k}{s^2+k^2}; s > 0$

$\mathscr{L}\{\cos(kt)\} = \frac{s}{s^2+k^2}; s > 0$

$\mathscr{L}\{\sinh(kt)\} = \frac{k}{s^2-k^2}; s > |k|$

$\mathscr{L}\{\cosh(kt)\} = \frac{s}{s^2-k^2}; s > |k|$

$\mathscr{L}\{u_a(t)\} = \frac{e^{-as}}{s};\ s > 0$

$\mathscr{L}\{\delta(t-t_0)\} = e^{-st_0}$

$\mathscr{L}\{\delta(t)\} = 1$

Inverse Laplace transforms of some basic functions

$\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} = 1$

$\mathscr{L}^{-1}\left\{\frac{1}{s^n}\right\} = \frac{1}{(n-1)!}t^{n-1}$

$\mathscr{L}^{-1}\left\{\frac{1}{s-k}\right\} = e^{kt}$

$\mathscr{L}^{-1}\left\{\frac{1}{s^2+k^2}\right\} = \frac{1}{k}\sin(kt)$

$\mathscr{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\} = \cos(kt)$

$\mathscr{L}^{-1}\left\{\frac{1}{s^2-k^2}\right\} = \frac{1}{k}\sinh(kt)$

$\mathscr{L}^{-1}\left\{\frac{s}{s^2-k^2}\right\} = \cosh(kt)$

$\mathscr{L}^{-1}\left\{\frac{e^{-as}}{s}\right\} = u_a(t)$

$\mathscr{L}^{-1}\left\{e^{-st_0}\right\} = \delta(t-t_0)$

$\mathscr{L}^{-1}\left\{1\right\} = \delta(t)$

Properties of the Laplace and Inverse Laplace transform

Translation Theorem I:

$$\mathscr{L}\left\{e^{kt}f(t)\right\} = F(s-k) = \mathscr{L}\{f(t)\}|_{s\to s-k}$$

$$\mathscr{L}^{-1}\{F(s-k)\} = \mathscr{L}^{-1}\left\{F(s)|_{s\to s-k}\right\} = e^{kt}\mathscr{L}^{-1}\{F(s)\} = e^{kt}f(t)$$

Translation Theorem II:

$$\mathscr{L}\left\{f(t-a)u_a(t)\right\} = e^{-as}F(s) = e^{-as}\mathscr{L}\{f(t)\}$$

$$\mathscr{L}^{-1}\{e^{-as}F(s)\} = f(t-a)u_a(t) = \mathscr{L}^{-1}\{F(s)\}\big|_{t\to t-a}\,u_a(t)$$

Derivatives of Laplace Transforms:

$$\mathscr{L}\left\{t^n f(t)\right\} = (-1)^n\frac{d^n}{ds^n}F(s)$$

$$\mathscr{L}^{-1}\left\{F^{(n)}(s)\right\} = (-1)^n t^n f(t)$$

Laplace Transform of Periodic Functions:
If $f(t)$ is piecewise continuous on $[0,\infty)$, of exponential order, and periodic with period T:

$$\mathscr{L}\{f(t)\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)\,dt$$

Laplace Transforms of Derivatives:

$$\begin{aligned}
\mathscr{L}\{y'\} &= sY(s)-y(0)\\
\mathscr{L}\{y''\} &= s^2Y(s)-sy(0)-y'(0)\\
&\vdots\\
\mathscr{L}\{y^{(n)}(t)\} &= s^nY(s)-s^{n-1}y(0)-s^{n-2}y'(0)-\ldots-sy^{(n-2)}(0)-y^{(n-1)}(0)
\end{aligned}$$

Convolution Theorem:

$$\mathscr{L}\{f(t)\star g(t)\} = F(s)G(s) = \mathscr{L}\{f(t)\}\mathscr{L}\{g(t)\}$$

$$\mathscr{L}^{-1}\{F(s)G(s)\} = f(t)\star g(t)$$

Dirac Delta Function:

$$\int_0^\infty f(t)\delta(t-t_0)\,dt = f(t_0)$$

$$(f\star\delta)(t) = f(t)$$

1. Consider the linear equation:

$$y^{(5)} + 5y^{(4)} - 2y''' - 10y'' + y' + 5y = g(x).$$

The complementary solution is:

$$y_c = c_1 e^{-5x} + c_2 e^{x} + c_3 x e^{x} + c_4 e^{-x} + c_5 x e^{-x}.$$

For each of the functions $g(x)$ below, write down the correct guess for a particular solution y_p, if one were to use the method of Undetermined Coefficients to solve the equation:

a). $g(x) = e^{3x}$

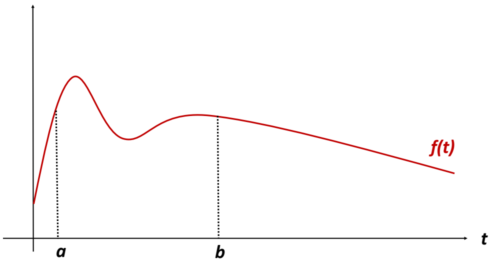
b). $g(x) = e^{x}$

c). $g(x) = 3e^{-5x} + \cos(2x)$

d). $g(x) = xe^{x}$

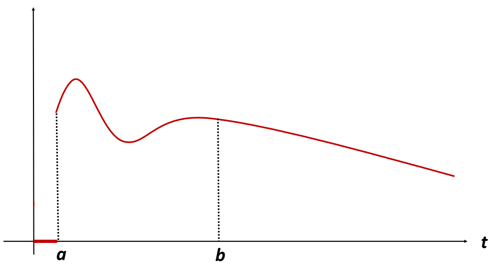
e). $g(x) = \sin(2x)e^{-5x}$.

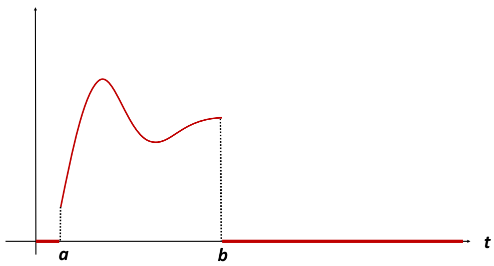
2. Consider the function $f(t)$, for $t \in [0, \infty)$, graphed below, and points $a, b \in [0, \infty)$.

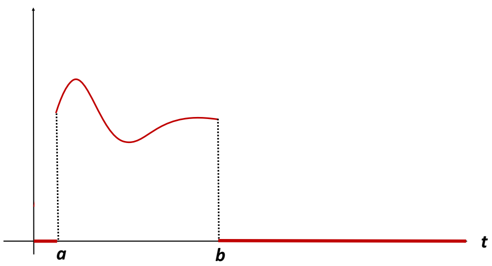


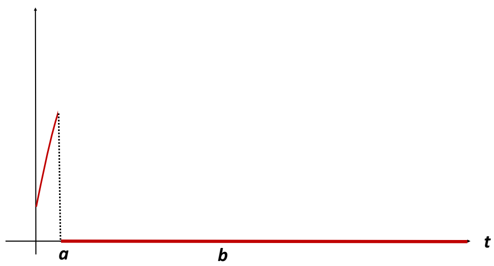
Match each of the following graphs (obtained by various translations and "turning off" of the graph of f) with the expressions below:

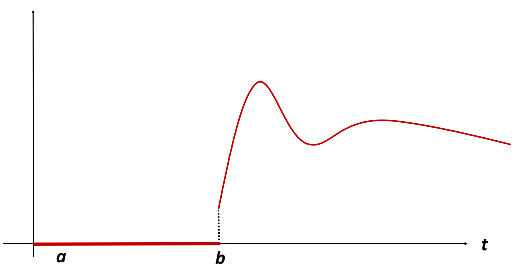
1. $f(t)(1 - u_b(t))$
2. $f(t)(u_a(t) - u_b(t))$
3. $f(t)(1 - u_a(t))$
4. $f(t)u_a(t)$
5. $f(t-a)(u_a(t) - u_b(t))$
6. $f(t-b)u_b(t)$

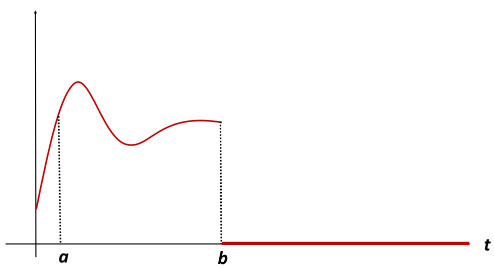












3. Find:

a). $\mathscr{L}\{t(e^t + e^{2t})^2\}$

b). $\mathscr{L}\{te^{2t}\sin(6t)\}$

c). $\mathscr{L}^{-1}\left\{\dfrac{s}{(s-2)(s-3)(s-6)}\right\}$

d). $\mathscr{L}^{-1}\left\{\dfrac{s}{s^2+4s+5}\right\}$

e). $\mathscr{L}^{-1}\left\{\dfrac{se^{-3s}}{s^2+4}\right\}$

f). $\mathscr{L}\{f(t)\}$, where $f(t)=\begin{cases} 2 & ,0\le t<1 \\ t & ,1\le t<3 \\ t^2 & ,t\ge 3. \end{cases}$

4. Find the Laplace transform $\mathscr{L}\{f(t)\}$ of the function $f(t)$, defined as

$$f(t) = 2(t-k), \text{ for all } t \in [k, k+1), \text{ for all integers } k \geq 0.$$

5. Solve the differential equation:

$$y'' + 3y' + 2y = \frac{1}{1 + e^x}.$$

6. Solve the differential equation:

$$x^2y'' - xy' + y = 4x \ln x.$$

7. Find a homogeneous linear differential equation with constant coefficients whose solution could be $4e^{6x} + \pi e^{-2x}$.

8. Suppose $y(t)$ is the solution to the initial value problem:

$$y'' + 5y' + 3y = 0;\ \ y(0) = 1,\ y'(0) = -1.$$

Find $Y(s) = \mathscr{L}\{y(t)\}$.

9. Find the general solution to the equation

$$y^{(6)} + 8y''' = 0.$$