1. Consider the linear equation:

$$y^{(5)} + 5y^{(4)} - 2y''' - 10y'' + y' + 5y = g(x).$$

The complementary solution is:

$$y_c = c_1e^{-5x} + c_2e^{x} + c_3xe^{x} + c_4e^{-x} + c_5xe^{-x}.$$

For each of the functions $g(x)$ below, write down the correct guess for a particular solution y_p, if one were to use the method of Undetermined Coefficients to solve the equation:

a). $g(x) = e^{3x}$

$$y_p = Ae^{3x}$$

b). $g(x) = e^{x}$

$$y_p = Ax^2e^x$$

c). $g(x) = 3e^{-5x} + \cos(2x)$

$$y_p = Axe^{-5x} + B\cos(2x) + C\sin(2x)$$

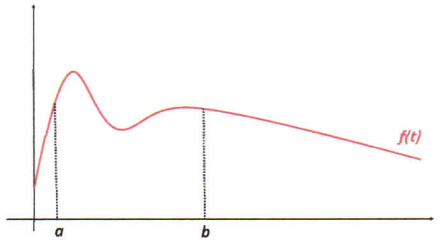
d). $g(x) = xe^{x}$

$$y_p = (Ax+B)e^x \cdot x^2 = (Ax^3 + Bx^2)e^x$$

e). $g(x) = \sin(2x)e^{-5x}$.

$$y_p = A\sin(2x)e^{-5x} + B\cos(2x)e^{-5x}.$$

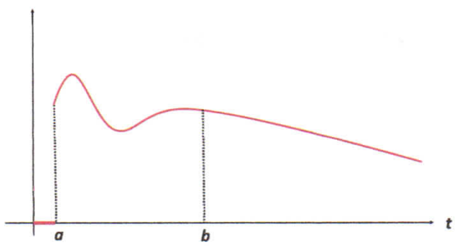
2. Consider the function $f(t)$, for $t \in [0, \infty)$, graphed below, and points $a, b \in [0, \infty)$.

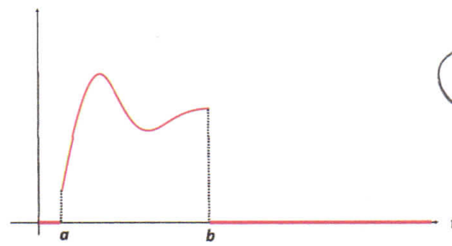


Match each of the following graphs (obtained by various translations and "turning off" of the graph of f) with the expressions below:

1. $f(t)(1 - u_b(t))$
2. $f(t)(u_a(t) - u_b(t))$
3. $f(t)(1 - u_a(t))$
4. $f(t)u_a(t)$
5. $f(t-a)(u_a(t) - u_b(t))$
6. $f(t-b)u_b(t)$

4





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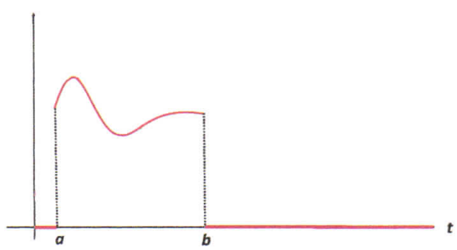
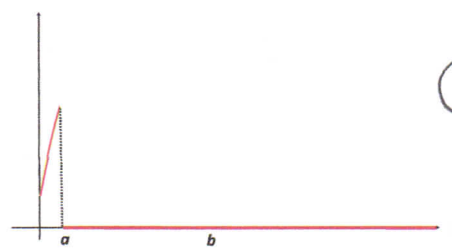
2





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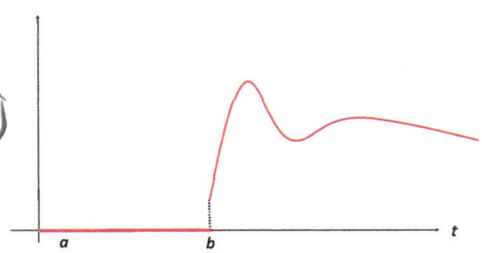
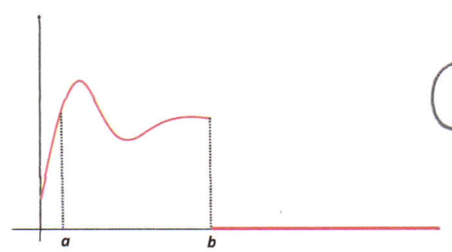
6





1

3. Find:

a). $\mathscr{L}\{t(e^t + e^{2t})^2\} = \mathcal{L}\{t(e^{2t} + 2e^{3t} + e^{4t})\}$

$$= \mathcal{L}\{t\}\big|_{s\to s-2} + 2\mathcal{L}\{t\}\big|_{s\to s-3} + \mathcal{L}\{t\}\big|_{s\to s-4}$$

$$= \frac{1}{(s-2)^2} + \frac{2}{(s-3)^2} + \frac{1}{(s-4)^2}$$

b). $\mathscr{L}\{te^{2t}\sin(6t)\} = -\frac{d}{ds}\mathcal{L}\{e^{2t}\sin(6t)\}$

$$= -\frac{d}{ds}\mathcal{L}\{\sin(6t)\}\big|_{s\to s-2}$$

$$= -\frac{d}{ds}\frac{6}{(s-2)^2+36} = \frac{12(s-2)}{((s-2)^2+36)^2}$$

c). $\mathscr{L}^{-1}\left\{\frac{s}{(s-2)(s-3)(s-6)}\right\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-6}\right\}$

$$= \frac{1}{2}e^{2t} - e^{3t} + \frac{1}{2}e^{6t}$$

$$\frac{s}{(s-2)(s-3)(s-6)} = \frac{A}{s-2} + \frac{B}{s-3} + \frac{C}{s-6}$$

$$\frac{s}{(s-3)(s-6)}\bigg|_{s=2} = A \Rightarrow A = 1/2$$

$$\frac{s}{(s-2)(s-6)}\bigg|_{s=3} = B \Rightarrow B = -1$$

$$\frac{s}{(s-2)(s-3)}\bigg|_{s=6} = C \Rightarrow C = \frac{1}{2}$$

d). $\mathscr{L}^{-1}\left\{\frac{s}{s^2+4s+5}\right\} = \mathscr{L}^{-1}\left\{\frac{s}{(s+2)^2+1}\right\} = \mathscr{L}^{-1}\left\{\frac{s+2-2}{(s+2)^2+1}\right\}$

$= \mathscr{L}^{-1}\left\{\frac{s}{s^2+1}\Big|_{s\to s+2}\right\} - 2\mathscr{L}^{-1}\left\{\frac{1}{s^2+1}\Big|_{s\to s+2}\right\}$

$= e^{-2t}\cos t - 2e^{-2t}\sin t.$

e). $\mathscr{L}^{-1}\left\{\frac{se^{-3s}}{s^2+4}\right\} = \mathscr{L}^{-1}\left\{\frac{s}{s^2+4}\right\}\Big|_{t\to t-3} u_3(t) = \cos(2t-6)\,u_3(t).$

f). $\mathscr{L}\{f(t)\}$, where $f(t) = \begin{cases} 2 & ,0 \le t < 1 \\ t & ,1 \le t < 3 \\ t^2 & ,t \ge 3. \end{cases} = 2\left(1-u_1(t)\right) + t\left(u_1(t) - u_3(t)\right) + t^2 u_3(t)$

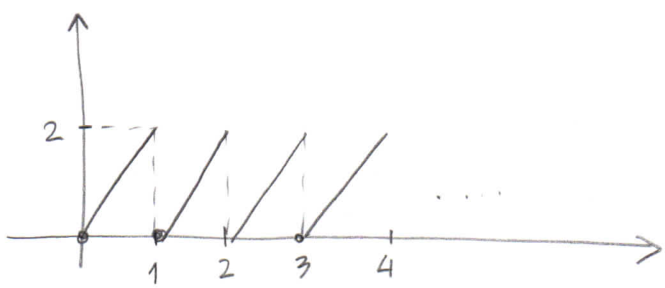
$$\mathscr{L}\{f(t)\} = \mathscr{L}\left\{2 - 2u_1(t) + (t-1+1)u_1(t) - (t-3+3)u_3(t) + ((t-3)+3)^2 u_3(t)\right\}$$

$$= \frac{2}{s} - 2\frac{e^{-s}}{s} + e^{-s}\mathscr{L}\{t+1\} - e^{-3s}\mathscr{L}\{t+3\} + e^{-3s}\mathscr{L}\{(t+3)^2\}$$

$$= \frac{2}{s} - 2\frac{e^{-s}}{s} + e^{-s}\left(\frac{1}{s^2} + \frac{1}{s}\right) - e^{-3s}\left(\frac{1}{s^2} + \frac{3}{s}\right) + e^{-3s}\left(\frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s}\right).$$

4. Find the Laplace transform $\mathscr{L}\{f(t)\}$ of the function $f(t)$, defined as

$$f(t) = 2(t-k), \text{ for all } t \in [k, k+1), \text{ for all integers } k \geq 0.$$



$$f(t) = \begin{cases} 2t, & t \in [0,1) \\ 2(t-1), & t \in [1,2) \\ 2(t-2), & t \in [2,3) \\ \vdots & \end{cases}$$

f = periodic w/ period $T = 1$

$$\mathscr{L}\{f(t)\} = \frac{1}{1-e^{-s}} \int_0^1 e^{-st} f(t)\, dt$$

$$= \frac{1}{1-e^{-s}} \mathscr{L}\left\{2t\,(1-u_1(t))\right\}$$

$$= \frac{2}{1-e^{-s}} \mathscr{L}\{t - (t-1+1)u_1(t)\}$$

$$= \frac{2}{1-e^{-s}} \left(\frac{1}{s^2} - e^{-s}\mathscr{L}\{t+1\}\right)$$

$$= \frac{2}{1-e^{-s}} \left(\frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-s}}{s}\right)$$

5. Solve the differential equation:

$$y'' + 3y' + 2y = \frac{1}{1+e^x}.$$

y_c : $m^2+3m+2=0$

$(m+2)(m+1)=0 \Rightarrow m=-2,-1 \Rightarrow \boxed{y_c = c_1e^{-2x}+c_2e^{-x}}$

y_p : $W = \begin{vmatrix} e^{-2x} & e^{-x} \\ -2e^{-2x} & -e^{-x} \end{vmatrix} = e^{-3x}$

$W_1 = \begin{vmatrix} 0 & e^{-x} \\ \frac{1}{1+e^x} & -e^{-x} \end{vmatrix} = -\frac{e^{-x}}{1+e^x} \Rightarrow u_1' = -\frac{e^{2x}}{1+e^x}$

$= -\frac{e^x(1+e^x-1)}{1+e^x}$

$= -e^x + \frac{e^x}{1+e^x} \Rightarrow \boxed{u_1 = -e^x + \ln(1+e^x)}$

$W_2 = \begin{vmatrix} e^{-2x} & 0 \\ -2e^{-2x} & \frac{1}{1+e^x} \end{vmatrix} = \frac{e^{-2x}}{1+e^x} \Rightarrow u_2' = \frac{e^x}{1+e^x} \Rightarrow \boxed{u_2 = \ln(1+e^x)}$

$\Rightarrow y_p = e^{-2x}(-e^x+\ln(1+e^x)) + e^{-x}\ln(1+e^x)$

$$\boxed{y = c_1e^{-2x}+c_2e^{-x}+e^{-2x}\ln(1+e^x)+e^{-x}\ln(1+e^x)}$$

6. Solve the differential equation:

$$x^2y'' - xy' + y = 4x\ln x.$$

$$y'' - \frac{1}{x}y' + \frac{1}{x^2}y = \frac{4}{x}\ln x$$

y_c Cauchy-Euler: $x^2y'' - xy' + y = 0$; $a=1, b=-1, c=1$

Char. Eq.: $m^2 - 2m + 1 = 0$; $(m-1)^2 = 0 \Rightarrow m_1 = m_2 = 1$

$$\boxed{y_c = C_1x + C_2x\ln x}$$

y_p

$$W = \begin{vmatrix} x & x\ln x \\ 1 & 1+\ln x \end{vmatrix} = x$$

$$W_1 = \begin{vmatrix} 0 & x\ln x \\ \frac{4}{x}\ln x & 1+\ln x \end{vmatrix} = -4\ln^2 x \Rightarrow u_1' = -\frac{4}{x}\ln^2 x \Rightarrow u_1 = -\frac{4}{3}\ln^3 x$$

$$W_2 = \begin{vmatrix} x & 0 \\ 1 & \frac{4}{x}\ln x \end{vmatrix} = 4\ln x \Rightarrow u_2' = \frac{4}{x}\ln x \Rightarrow u_2 = 2\ln^2 x$$

$$\Rightarrow \boxed{y_p = -\frac{4}{3}x\ln^3 x + 2x\ln^3 x = \frac{2}{3}x\ln^3 x}$$

$$\boxed{\Rightarrow y = C_1x + C_2x\ln x + \frac{2}{3}x\ln^3 x.}$$

7. Find a homogeneous linear differential equation with constant coefficients whose solution could be $4e^{6x} + \pi e^{-2x}$.

Roots of characteristic equation: $6\ ;\ -2$

$$(m-6)(m+2) = m^2 - 4m - 12$$

$$\boxed{y'' - 4y' - 12y = 0}$$

8. Suppose $y(t)$ is the solution to the initial value problem:

$$y'' + 5y' + 3y = 0;\ \ y(0) = 1,\ y'(0) = -1.$$

Find $Y(s) = \mathscr{L}\{y(t)\}$.

$$s^2Y(s) - s + 1 + 5sY(s) - 5 + 3Y(s) = 0$$

$$(s^2+5s+3)Y(s) = s+4$$

$$Y(s) = \frac{s+4}{s^2+5s+3}$$

9. Find the general solution to the equation

$$y^{(6)} + 8y''' = 0.$$

Char. Eqn.: $m^6 + 8m^3 = 0$

$m^3(m^3+8) = 0 \Rightarrow m^3(m+2)(m^2-2m+4) = 0$

$m_1 = m_2 = m_3 = 0$

$m_4 = -2$

$\Delta = 4 - 16 = -12$

$m_{5,6} = \frac{2 \pm 2\sqrt{3}i}{2} = 1 \pm i\sqrt{3}$

$$\boxed{y = C_1 + C_2x + C_3x^2 + C_4e^{-2x} + C_5e^x\cos(\sqrt{3}x) + C_6e^x\sin(\sqrt{3}x)}$$