Name: Solutions

September 23$^{\text{rd}}$, 2015.
Math 2552; Sections F1 – F4; L1 – L4.
Georgia Institute of Technology
Sample Exam 1

I commit to uphold the ideals of honor and integrity by refusing to betray the trust bestowed upon me as a member of the Georgia Tech community. By signing my name below I pledge that I have neither given nor received help on this exam.

Pledged: ____________________

Problem	Possible Score	Earned Score
1	16	
2	17	
3	17	
4	17	
5	17	
6	16	
Total	100	

Remember that you must SHOW YOUR WORK to receive credit!

Good luck!

1. Consider the autonomous equation:

$$\frac{dy}{dx} = (y-1)(y-3).$$

(a). Find the equilibrium solutions:

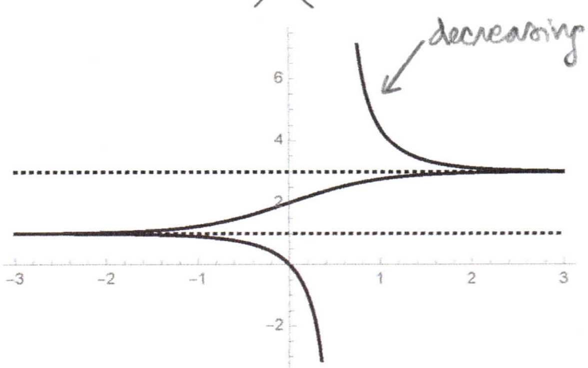
$y=1;\ y=3$

(b). Draw the phase portrait.

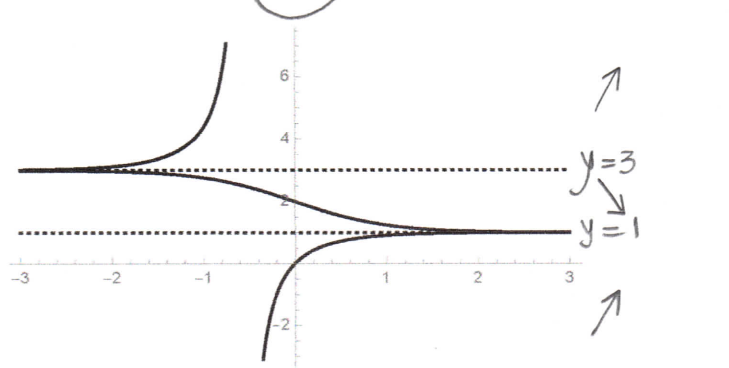
y		1		3	
$y-1$	$-$	0	$+$	$+$	$+$
$y-3$	$-$	$-$	$-$	0	$+$
$(y-1)(y-3)$	$+$	0	$-$	0	$+$

(c). Determine which of the graphs below could be possible solutions to this equation (circle the correct one).

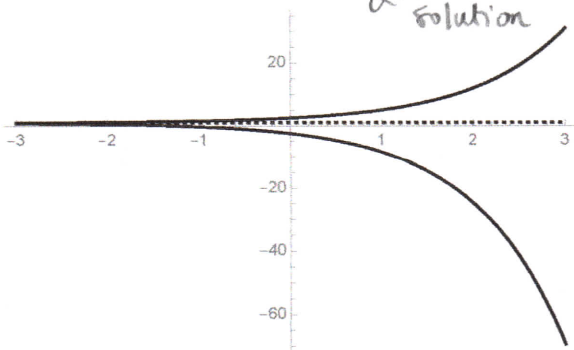
~~**A.**~~



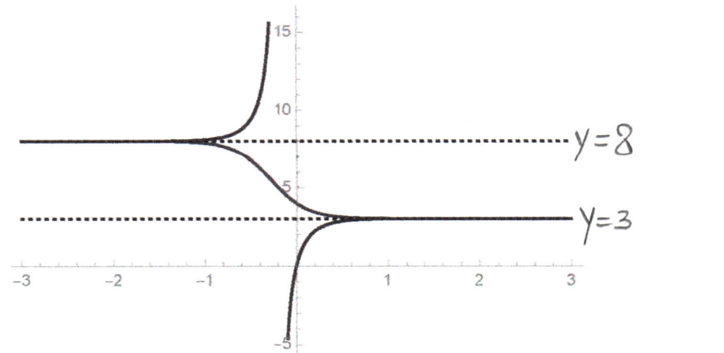
B.



~~**C.**~~ ← only one equilibrium solution



~~**D.**~~ ← wrong equilibrium solutions



2. Find an *explicit* solution to the initial value problem:

$$\frac{dy}{dx} = e^x + y; \quad y(0) = 0,$$

and state the largest interval where your solution is valid.

Linear First-Order ODE

Standard Form : $\dfrac{dy}{dx} - y = e^x$

Integrating Factor : $p(x) = -1$

$$\int p(x)\,dx = -x$$

$$\boxed{\mu(x) = e^{-x}}$$

Multiply (standard form) equation by $\mu(x)$:

$$\frac{d}{dx}\left(ye^{-x}\right) = 1$$

$$ye^{-x} = \int 1\,dx$$

$$ye^{-x} = x + C$$

$$y = e^x(x+C)$$

$$y(0) = 0 \implies 0 = e^0(0+C)$$

$$0 = C$$

$$\implies \boxed{y = xe^x} \quad \boxed{\begin{array}{l} x \in (-\infty, \infty) \\ \text{or } x \in \mathbb{R} \end{array}}$$

3. Consider the differential equation:

$$(y^2 - x^2)\, dx - 2xy\, dy = 0.$$

(a). Determine whether or not the equation is exact.

$$\left.\begin{array}{l} M_y = 2y \\ N_x = -2y \end{array}\right\} \Rightarrow M_y \neq N_x \Rightarrow \text{the equation is } \underline{\text{not}} \text{ exact}$$

(b). Solve the equation (give an *implicit* solution).

<u>Homogeneous</u> of degree 2.

$$\left(\frac{y^2}{x^2} - 1\right) - 2\frac{y}{x}\frac{dy}{dx} = 0$$

$$\boxed{u = \frac{y}{x}} \Rightarrow \boxed{\frac{dy}{dx} = u + x\frac{du}{dx}}$$

$$u^2 - 1 - 2u\left(u + x\frac{du}{dx}\right) = 0$$

$$-u^2 - 1 - 2ux\frac{du}{dx} = 0$$

$$-2ux\frac{du}{dx} = u^2 + 1$$

$$-\frac{2u}{u^2+1}du = \frac{1}{x}dx$$

$$-\ln(u^2+1) = \ln|x| + C$$

$$\frac{1}{u^2+1} = Cx$$

$$\frac{1}{\frac{y^2}{x^2}+1} = Cx$$

$$\boxed{\frac{x^2}{x^2+y^2} = Cx}$$

or $$\frac{x^2}{Cx} = x^2 + y^2$$

$$\boxed{x^2 + y^2 = Cx}$$

$$\boxed{\begin{array}{l} y = ux \\ dy = u\,dx + x\,du \end{array}}$$

$$(u^2x^2 - x^2)dx - 2x^2u(u\,dx + x\,du) = 0$$

$$(u^2 - 1)dx - 2u(u\,dx + x\,du) = 0$$

$$(-u^2 - 1)dx - 2ux\,du = 0$$

$$\boxed{\begin{array}{l} x = uy \\ dx = u\,dy + y\,du \end{array}}$$

$$(y^2 - u^2y^2)(u\,dy + y\,du) - 2uy^2dy = 0$$

$$(1-u^2)(u\,dy + y\,du) - 2u\,dy = 0$$

$$(u - u^3 - 2u)\,dy + (1-u^2)y\,du = 0$$

$$(1-u^2)y\frac{du}{dy} = u^3 + u$$

$$\frac{1-u^2}{u(1+u^2)}du = \frac{1}{y}dy$$

$$C + \ln|y| = \int \frac{1+u^2-2u^2}{u(1+u^2)}du = \ln|u| - \int\frac{2u}{1+u^2}du$$
$$= \ln|u| - \ln(1+u^2)$$
$$= \ln\left|\frac{u}{1+u^2}\right|$$

$$\frac{u}{1+u^2} = Cy$$

$$\frac{\frac{y}{x}}{\frac{x^2+y^2}{x^2}} = Cy \;;\; \frac{xy}{x^2+y^2} = Cy \;;\; \boxed{x^2+y^2 = Cx}$$

4. Suppose that $y(x)$ is the solution to the initial value problem:

$$\frac{1}{x}e^{y^2}\frac{dy}{dx} = \frac{1}{y(1+x^2)}; \quad y(0) = 0.$$

Find

$$e^{y^2(1)}.$$

Solve ODE : Separable :

$$ye^{y^2}dy = \frac{x}{1+x^2}dx$$

$$\int ye^{y^2}dy = \int \frac{x}{1+x^2}dx$$

$$\frac{1}{2}e^{y^2} = \frac{1}{2}\ln(1+x^2)+C$$

$$\boxed{e^{y^2} = \ln(1+x^2)+C}$$

Solve IVP : $y(0)=0 \Rightarrow e^0 = \ln(1+0)+C \Rightarrow C=1$

$$\boxed{e^{y^2} = \ln(1+x^2)+1}$$

Find $e^{y^2(1)}$: $e^{y^2(1)} = \ln(1+1)+1 \Rightarrow \boxed{e^{y^2(1)} = \ln(2)+1}$

5. Solve the differential equation:

$$y' = \frac{2 + ye^{xy}}{2y - xe^{xy}}.$$

Give an *implicit* solution.

$$\frac{dy}{dx} = \frac{2+ye^{xy}}{2y-xe^{xy}}$$

$$(2y-xe^{xy})\,dy = (2+ye^{xy})\,dx$$

$$(2+ye^{xy})\,dx + (xe^{xy}-2y)\,dy = 0$$

$$\left.\begin{aligned} M_y &= e^{xy} + xye^{xy} \\ N_x &= e^{xy} + xy\,e^{xy} \end{aligned}\right\} \Rightarrow \underline{\underline{\text{exact}}}$$

<u>Potential</u> : $\frac{\partial f}{\partial x} = 2+ye^{xy} \Rightarrow f(x,y) = 2x + e^{xy} + g(y)$

$$\Rightarrow \left.\begin{aligned} \frac{\partial f}{\partial y} &= xe^{xy} + g'(y) \\ &= xe^{xy} - 2y \end{aligned}\right\} \Rightarrow \begin{aligned} g'(y) &= -2y \\ g(y) &= -y^2 \end{aligned}$$

$$f(x,y) = 2x + e^{xy} - y^2$$

$$\boxed{2x + e^{xy} - y^2 = c}$$

6. Given that the general solution to the differential equation:

$$\boxed{\frac{du}{dt} - \frac{2}{t}u = 8t^2 \cos t}$$

is:

$$u(t) = 8t^2 \sin t + ct^2,$$

find the solution to the differential equation:

$$y\frac{dy}{dx} - \frac{1}{x}y^2 = 4x^2 \cos x.$$

$$\frac{dy}{dx} - \frac{1}{x}y = 4x^2\cos x \cdot \frac{1}{y}$$

<u>Bernoulli</u> w/ $\alpha = -1$

$1 - \alpha = 2$

$$\boxed{u = y^2}$$

$$\Rightarrow \boxed{\frac{du}{dx} - \frac{2}{x}u = 8x^2\cos x} \Rightarrow u = 8x^2 \sin x + cx^2$$

$$\Rightarrow \boxed{y^2 = 8x^2 \sin x + cx^2}$$