(1) $x^2y'' - 7xy' + 16y = 0$; $y_1 = x^4$

$y = ux^4$

$y' = u'x^4 + 4x^3u$

$y'' = u''x^4 + 8x^3u' + 12x^2u$

$$x^2y'' - 7xy' + 16y = x^6u'' + 8x^5u' + \cancel{12x^4u}$$
$$- 7x^5u' - \cancel{28x^4u}$$
$$+ \cancel{16x^4u}$$

$\Rightarrow \boxed{x^6u'' + x^5u' = 0}$ $w = u'$

$x^6w' + x^5w = 0$

$w' + \frac{1}{x}w = 0$

$\mu(x) = x \Rightarrow \frac{d}{dx}(wx) = 0 \Rightarrow w = \frac{c_1}{x} = u' \Rightarrow \boxed{u = c_1\ln|x| + c_2}$

$\Rightarrow \boxed{y = c_1x^4\ln|x| + c_2x^4}$; $y_2 = x^4\ln|x|$

(2) $xy'' + y' = 0$; $y_1 = \ln x$.

$y = u\ln x$

$y' = u'\ln x + \frac{u}{x}$

$y'' = u''\ln x + \frac{2u'}{x} - \frac{1}{x^2}u$

$$xy'' + y' = (x\ln x)u'' + 2u' - \cancel{\frac{1}{x}u}$$
$$+ (\ln x)u' + \cancel{\frac{1}{x}u}$$

$\Rightarrow \boxed{(x\ln x)u'' + (2 + \ln x)u' = 0}$ $w = u'$

$(x\ln x)w' + (2+\ln x)w = 0$

$w' + \frac{2+\ln x}{x\ln x}w = 0$

$\int p(x)\,dx = 2\int\frac{1}{x\ln x}dx + \int\frac{1}{x}dx = 2\ln|\ln x| + \ln x$

$\Rightarrow \boxed{\mu(x) = x\ln^2 x} \Rightarrow \frac{d}{dx}\left(w\,x\ln^2 x\right) = 0 \Rightarrow w = u' = \frac{c_1}{x\ln^2 x}$

$\Rightarrow u = \int\frac{c_1}{x\ln^2 x}dx = -\frac{c_1}{\ln x} + c_2 \Rightarrow \boxed{u = \frac{c_1}{\ln x} + c_2}$

$\Rightarrow \boxed{y = c_1 + c_2\ln x}$; $\boxed{y_2 = 1}$ (or any constant).

③ $(1-2x-x^2)\,y'' + 2(1+x)\,y' - 2y = 0\,; \quad y_1 = x+1.$

$y = u(x+1)$

$y' = u'(x+1) + u$

$y'' = u''(x+1) + 2u'$

$$(1-2x-x^2)y'' + 2(1+x)y' - 2y = u''(x+1)(1-2x-x^2) + 2(1-2x-x^2)u' + 2(x+1)^2u' + \cancel{2(x+1)u} \cancel{-2(x+1)u}$$

$$= u''(x+1)(1-2x-x^2) + 2(1\cancel{-2x}\cancel{-x^2} + 1\cancel{+2x}\cancel{+x^2})u'$$

$$\Rightarrow \boxed{u''(x+1)(1-2x-x^2) + 4u' = 0} \qquad w = u'$$

$$(x+1)(1-2x-x^2)w' + 4w = 0$$

$$w' + \frac{4}{(x+1)(1-2x-x^2)}w = 0$$

$$\frac{4}{(x+1)(1-2x-x^2)} = \frac{2\left[(x+1)^2 + (1-2x-x^2)\right]}{(x+1)(1-2x-x^2)}$$

$$= \frac{2x+2}{1-2x-x^2} + \frac{2}{x+1}$$

$$\Rightarrow \int p(x)\,dx = -\ln|1-2x-x^2| + 2\ln|x+1|$$

$$\Rightarrow \mu(x) = \frac{(x+1)^2}{1-2x-x^2} \Rightarrow \frac{d}{dx}\left(w\mu(x)\right) = 0$$

$$\Rightarrow w = u' = \frac{C_1(1-2x-x^2)}{(x+1)^2}$$

$$\Rightarrow u = C_1\int \frac{1-2x-x^2}{(x+1)^2}dx = C_1\int \frac{2-(x+1)^2}{(x+1)^2}dx$$

$$= -2C_1\frac{1}{x+1} - C_1x + C_2$$

$$\Rightarrow u = C_1\left(x + \frac{2}{x+1}\right) + C_2 = C_1\frac{x^2+x+2}{x+1} + C_2$$

$$\Rightarrow \boxed{y = C_1(x^2+x+2) + C_2(x+1)} \qquad y_2 = x^2+x+2.$$

④ $x^2y'' - xy' + 2y = 0$; $y_1 = x\sin(\ln x)$.

$$y = u\,x\sin(\ln x)$$

$$y' = u'x\sin(\ln x) + u\left(\sin(\ln x) + \cos(\ln x)\right)$$

$$y'' = u''x\sin(\ln x) + 2u'\left(\sin(\ln x) + \cos(\ln x)\right) + u\left(\cos(\ln x)\cdot\frac{1}{x} - \sin(\ln x)\cdot\frac{1}{x}\right)$$

$$\Rightarrow x^2y'' - xy' + 2y = u''x^3\sin(\ln x) + 2x^2\left(\sin(\ln x) + \cos(\ln x)\right)u' + u\left(x\cancel{\cos(\ln x)} - x\cancel{\sin(\ln x)}\right) - x^2\sin(\ln x)u' - u\left(x\cancel{\cos(\ln x)} + x\cancel{\sin(\ln x)}\right) + 2ux\cancel{\sin(\ln x)}$$

$$= u''x^3\sin(\ln x) + \left(x^2\sin(\ln x) + 2x^2\cos(\ln x)\right)u'$$

$w = u'$: $$w' + \frac{x^2\sin(\ln x) + 2x^2\cos(\ln x)}{x^3\sin(\ln x)}\,w = 0$$

$$w' + \left(\frac{1}{x} + \frac{2\cos(\ln x)}{x\sin(\ln x)}\right)w = 0$$

$$\int p(x)\,dx = \ln x + 2\ln|\sin(\ln x)| \Rightarrow \mu(x) = x\sin^2(\ln x)$$

$$\Rightarrow \frac{d}{dx}\left(w\mu(x)\right) = 0 \Rightarrow w = \frac{c_1}{x\sin^2(\ln x)} = u'$$

$$\Rightarrow u = -c_1\cot(\ln x) + c_2$$

$$\boxed{u = c_1\cot(\ln x) + c_2} \Rightarrow \boxed{y = c_1x\cos(\ln x) + c_2x\sin(\ln x)}$$

$$\boxed{y_2 = x\cos(\ln x)}$$

5 $(3x+1)y'' - (9x+6)y' + 9y = 0$; $y_1 = e^{3x}$

$y = ue^{3x}$

$y' = u'e^{3x} + 3ue^{3x}$

$y'' = u''e^{3x} + 6u'e^{3x} + 9ue^{3x}$

$$(3x+1)y'' - (9x+6)y' + 9y = (3x+1)e^{3x}u'' + 6(3x+1)e^{3x}u' + 9(3x+1)e^{3x}u$$
$$- (9x+6)e^{3x}u' - 3(9x+6)e^{3x}u$$
$$+ 9e^{3x}u$$

$$(3x+1)e^{3x}u'' + (18x+6-9x-6)e^{3x}u' + (\cancel{27x}+\cancel{9}-\cancel{27x}-\cancel{18}+\cancel{9})e^{3x}u$$

$\Rightarrow (3x+1)e^{3x}u'' + 9xe^{3x}u' = 0$

$\Rightarrow (3x+1)u'' + 9xu' = 0$ $\quad w = u'$

$\Rightarrow (3x+1)w' + 9xw = 0 \Rightarrow w' + \frac{9x}{3x+1}w = 0$

$$\int \frac{9x}{3x+1}dx = 3\int \frac{3x+1-1}{3x+1}dx = 3\left(x - \frac{1}{3}\ln|3x+1|\right)$$
$$= 3x - \ln|3x+1|$$

$\Rightarrow \mu(x) = \frac{1}{3x+1}e^{3x}$

$\Rightarrow w = u' = \frac{c_1(3x+1)}{e^{3x}}$

$$\int \frac{3x+1}{e^{3x}} = 3\int xe^{-3x}dx + \int e^{-3x}dx$$
$$= -\int x(e^{-3x})'dx - \frac{1}{3}e^{-3x} + c_2$$
$$= -xe^{-3x} + \int e^{-3x}dx - \frac{1}{3}e^{-3x} + c_2$$
$$= -xe^{-3x} - \frac{2}{3}e^{-3x} + c_2$$

$\Rightarrow u = c_1\frac{-(3x+2)}{e^{3x}} + c_2$

$u = \frac{c_1(3x+2)}{e^{3x}} + c_2$

$\Rightarrow \boxed{y = c_1(3x+2) + c_2e^{3x}}$

$\boxed{y_2 = 3x+2}$

⑥ $x^2y'' - xy' + y = 0$; $y_1 = x$.

$y = ux$

$y' = u'x + u$

$y'' = u''x + 2u'$

$$x^2y'' - xy' + y = x^3u'' + 2x^2u' - x^2u' - \cancel{xu} + \cancel{xu}$$

$$\Rightarrow x^3u'' + x^2u' = 0$$

$w = u'$ $\Rightarrow$ $w' + \frac{1}{x}w = 0$

$$\Rightarrow \frac{d}{dx}(wx) = 0 \Rightarrow w = \frac{c_1}{x} = u' \Rightarrow u = c_1 \ln|x| + c_2$$

$$\Rightarrow \boxed{y = c_1 x \ln|x| + c_2 x} \; ; \; \boxed{y_2 = x \ln|x|}$$

⑦ $(1-x^2)y'' - 2xy' = 0$; $y_1 = 1$

$y = u$

$w = u' = y'$ $\Rightarrow$ $(1-x^2)w' - 2xw = 0$

$$w' - \frac{2x}{1-x^2}w = 0$$

$$\mu(x) = 1-x^2 \Rightarrow \frac{d}{dx}\left(w(1-x^2)\right) = 0 \Rightarrow w = \frac{c_1}{1-x^2} = y'$$

$$\Rightarrow y = c_1\int \frac{1}{1-x^2}dx = \frac{c_1}{2}\int \frac{(1-x)+(1+x)}{(1-x)(1+x)}$$

$$= \frac{c_1}{2} \ln\left|\frac{1+x}{1-x}\right| + c_2$$

$$\Rightarrow \boxed{y = c_1 \ln\left|\frac{1+x}{1-x}\right| + c_2} \; ; \; \boxed{y_2 = \ln\left|\frac{1+x}{1-x}\right|}$$

⑧ $4x^2y'' + y = 0 \; ; \; y_1 = \sqrt{x}\ln x$

$$y = u\sqrt{x}\ln x$$

$$y' = u'\sqrt{x}\ln x + u\left(\frac{1}{2\sqrt{x}}\ln x + \frac{1}{\sqrt{x}}\right)$$

$$y'' = u''\sqrt{x}\ln x + 2u'\left(\frac{1}{2\sqrt{x}}\ln x + \frac{1}{\sqrt{x}}\right) + u\left(\cancel{\frac{1}{2x\sqrt{x}}} - \frac{1}{4x\sqrt{x}}\ln x - \cancel{\frac{1}{2x\sqrt{x}}}\right)$$

$$4x^2y'' + y = \left(4x^2\sqrt{x}\ln x\right)u'' + \left(4x\sqrt{x}\ln x + 8x\sqrt{x}\right)u' \cancel{- \sqrt{x}\ln x\, u} \cancel{+ \sqrt{x}\ln x\, u}$$

$$w = u'$$

$$w' + \frac{4x\sqrt{x}\,(\ln x + 2)}{4x^2\sqrt{x}\ln x}\, w = 0$$

$$w' + \frac{\ln x + 2}{x\ln x}\, w = 0$$

$$\int \frac{\ln x + 2}{x\ln x}\, dx = \ln x + 2\ln(\ln x) = \ln\left(x\ln^2 x\right)$$

$$\Rightarrow \mu(x) = x\ln^2 x$$

$$\Rightarrow \frac{d}{dx}\left(w\, x\ln^2 x\right) = 0 \Rightarrow w = u' = \frac{c_1}{x\ln^2 x}$$

$$\Rightarrow \boxed{u = \frac{c_1}{\ln x} + c_2}$$

$$\boxed{y = c_1\sqrt{x} + c_2\sqrt{x}\ln x} \; ; \quad \boxed{y_2 = \sqrt{x}}$$