Math 2552 - Differential Equations **Homework 9**
Sections F1 – F4; L1 – L4
Georgia Institute of Technology, Fall 2015

Reduction of Order

Use reduction of order to find a second solution and the general solution to the differential equations below, given one of the solutions (on the right of each equation).

1. $x^2y'' - 7xy' + 16y = 0;\;\; y_1 = x^4.$

2. $xy'' + y' = 0;\;\; y_1 = \ln x.$

3. $(1 - 2x - x^2)y'' + 2(1 + x)y' - 2y = 0;\;\; y_1 = x + 1.$

4. $x^2y'' - xy' + 2y = 0;\;\; y_1 = x \sin(\ln x).$

5. $(3x + 1)y'' - (9x + 6)y' + 9y = 0;\;\; y_1 = e^{3x}.$

6. $x^2y'' - xy' + y = 0;\;\; y_1 = x.$

7. $(1 - x^2)y'' - 2xy' = 0;\;\; y_1 = 1.$

8. $4x^2y'' + y = 0;\;\; y_1 = \sqrt{x} \ln x.$

Reduction of Order

Start with a 2^{nd} order linear ODE

$$y'' + P(x)\,y' + Q(x)\,y = 0 \qquad (*)$$

and suppose y_1 is a Known solution.

(Assume we are working on an interval I where $y_1(x) \neq 0$ for all $x \in I$.)

Look for another solution, of the form $\boxed{y = u y_1}$ where u is some unknown function of x

$$y = u y_1 \Rightarrow y' = u' y_1 + u y_1'$$

$$\Rightarrow y'' = u'' y_1 + 2u' y_1' + u y_1''$$

$$\Rightarrow y'' + P(x) y' + Q(x) y = u'' y_1 + 2u' y_1' + u y_1'' + P(x) u' y_1 + P(x) u y_1' + Q(x) u y_1$$

$$= u \underbrace{\left(y_1'' + P(x) y_1' + Q(x) y_1\right)}_{0 \text{ b/c } y_1 \text{ is a solution to } (*)} + u'' y_1 + \left(2y_1' + P(x) y_1\right) u'$$

$\Rightarrow y = u y_1$ is a solution to $(*)$ if and only if

$$\boxed{y_1 u'' + (2y_1' + P(x) y_1) u' = 0}$$

Obtain an equation in u of the form:

$$y_1 u'' + (2y_1' + P(x) y_1)\, u' = 0$$

Make the substitution $\boxed{w = u'}$:

$$y_1 w' + (2y_1' + P(x) y_1) w = 0$$

(linear & separable)
Solve and find w

$$w = u' \Rightarrow y_1 w' + (2y_1' + P(x) y_1) w = 0$$

$$\Rightarrow \frac{dw}{dx} + \left(\frac{2y_1'}{y_1} + P(x)\right) w = 0$$

$$\Rightarrow \frac{1}{w}\,dw = -\left(\frac{2y_1'}{y_1} + P(x)\right) dx$$

$$\Rightarrow \ln|w| = -2\ln|y_1| - \int P(x)\,dx + C$$

$$\Rightarrow \ln|w y_1^2| = -\int P(x)\,dx + C$$

$$\Rightarrow w y_1^2 = C\, e^{-\int P(x)\,dx}$$

$$\Rightarrow \boxed{w = u' = C\,\frac{1}{y_1^2} e^{-\int P(x)\,dx}}$$

From $w = u'$, find u

$$\Rightarrow \boxed{u = C_2 \int \frac{1}{y_1^2} e^{-\int P(x)\,dx}\,dx + C_1}$$

$$\Rightarrow y = C_1 y_1 + C_2 \underbrace{y_1 \int \frac{1}{y_1^2} e^{-\int P(x)\,dx}\,dx}_{y_2 \text{ (new solution)}}$$

Finally, find $y = u y_1$:
You will obtain

$$\boxed{y = C_1 y_1 + C_2 y_2}$$

(the general solution)

(y_2 is another solution, and $\{y_1, y_2\}$ are linearly independent)

Example : Given that $\boxed{y=e^x}$ is a solution to $\boxed{y''-y=0}$, find the general solution.

- Look for a solution $y=ue^x$ for some unknown function of u.

$$y'=u'e^x+ue^x$$
$$y''=u''e^x+2u'e^x+ue^x$$

$\Rightarrow y''-y=u''e^x+2u'e^x=(u''+2u')e^x$

$\Rightarrow y=ue^x$ is a solution if and only if $\boxed{u''+2u'=0}$

(because $e^x \neq 0$ for all real x).

- Solve $\boxed{u''+2u'=0}$

Substitution: $\boxed{w=u'}$

$\Rightarrow \boxed{w'+2w=0}$ (Separable)

$\frac{dw}{dx}=-2w \Rightarrow \frac{1}{w}dw=-2dx$

$\Rightarrow \ln|w|=-2x+c \Rightarrow \boxed{w=ce^{-2x}}$

$\Big\downarrow w=u'$

$u'=ce^{-2x} \Rightarrow u=\int ce^{-2x}dx$

$$\boxed{u=-\tfrac{1}{2}ce^{-2x}+c_1}$$

- Find $y=ue^x$: $y=ue^x=-\frac{1}{2}ce^{-x}+c_1e^x$

$\Rightarrow \boxed{y=c_1e^x+c_2e^{-x}}$ General solution