Homework 8
- Solutions -

Method of Undetermined Coefficients

① $y'' + 3y' + 2y = 6$

Complementary Solution : $y'' + 3y' + 2y = 0$

$$m^2 + 3m + 2 = 0$$

$$(m+1)(m+2) = 0 \Rightarrow m = -1, -2 \qquad \boxed{y_c = c_1 e^{-x} + c_2 e^{-2x}}$$

Particular Solution : $y_p = A \Rightarrow y_p' = y_p'' = 0$

$$\Rightarrow 2A = 6 \Rightarrow A = 3 \qquad \boxed{y_p = 3}$$

General Solution : $\boxed{y = c_1 e^{-x} + c_2 e^{-2x} + 3}$

② $y'' + 4y = 3\sin(2x)$.

Complementary Solution : $y'' + 4y = 0$

$$m^2 + 4 = 0 \Rightarrow m = \pm 2i \qquad \boxed{y_c = c_1 \sin(2x) + c_2 \cos(2x)}$$

Particular Solution : Initial guess would be $y_p = A\sin(2x) + B\cos(2x)$
but these are contained in the complementary solution!

So take $y_p = Ax\sin(2x) + Bx\cos(2x)$

$$y_p' = A\sin(2x) + 2Ax\cos(2x) + B\cos(2x) - 2Bx\sin(2x)$$

$$y_p'' = 2A\cos(2x) + 2A\cos(2x) - 4Ax\sin(2x) - 2B\sin(2x)$$
$$- 2B\sin(2x) - 4Bx\cos(2x)$$
$$= 4A\cos(2x) - 4B\sin(2x) - 4Ax\sin(2x) - 4Bx\cos(2x)$$

$$\Rightarrow y_p'' + 4y_p = 4A\cos(2x) - 4B\sin(2x) + (\cancel{4A - 4A})x\sin(2x) + (\cancel{4B - 4B})x\cos(2x)$$
$$= 3\sin(2x)$$

$$\Rightarrow \begin{cases} 4A = 0 \\ -4B = 3 \end{cases} \Rightarrow A = 0;\ B = -3/4 \Rightarrow \boxed{y_p = -\frac{3}{4}x\cos(2x)}$$

General Solution : $\boxed{y = c_1\sin(2x) + c_2\cos(2x) - \frac{3}{4}x\cos(2x)}$

③ $y'' + 2y' + y = \sin x + 3\cos(2x)$

Complementary solution : $y'' + 2y' + y = 0$

$m^2 + 2m + 1 = 0$

$(m+1)^2 = 0$

$m = -1$ (double root)

$$\boxed{y_c = c_1 e^{-x} + c_2 x e^{-x}}$$

Particular Solution : $y_p = A\sin x + B\cos x + C\sin(2x) + D\cos(2x)$

$y_p' = A\cos x - B\sin x + 2C\cos(2x) - 2D\sin(2x)$

$y_p'' = -A\sin x - B\cos x - 4C\sin(2x) - 4D\cos(2x)$

$$\begin{aligned} y_p'' + 2y_p' + y_p = & -A\sin x - B\cos x - 4C\sin(2x) - 4D\cos(2x) \\ & -2B\sin x + 2A\cos x - 4D\sin(2x) + 4C\cos(2x) \\ & + A\sin x + B\cos x + C\sin(2x) + D\cos(2x) \\ \hline & -2B\sin x + 2A\cos x + (-3C-4D)\sin(2x) + (4C-3D)\cos(2x) \\ & = \sin x + 3\cos(2x) \end{aligned}$$

$$\begin{cases} -2B = 1 & \Rightarrow B = -1/2 \\ 2A = 0 & \Rightarrow A = 0 \\ -3C - 4D = 0 \\ 4C - 3D = 3 \end{cases}$$

$$\boxed{y_p = -\frac{1}{2}\cos x + \frac{12}{25}\sin(2x) - \frac{9}{25}\cos(2x)}$$

$$\begin{aligned} -12C - 16D &= 0 \\ 12C - 9D &= 9 \\ \hline / \quad -25D &= 9 \end{aligned}$$

$D = -9/25$

$C = \frac{3}{4}(D+1) = \frac{3}{4}\cdot\frac{16}{25} = \frac{12}{25}$

$$\boxed{y = c_1 e^{-x} + c_2 x e^{-x} - \frac{1}{2}\cos x + \frac{12}{25}\sin(2x) - \frac{9}{25}\cos(2x)}$$

4 $y'' - 10y' + 25y = 30x + 3$

Complementary Solution : $y'' - 10y' + 25y = 0$

$m^2 - 10m + 25 = 0$

$(m-5)^2 = 0$

$m = 5$ (double root)

$$\boxed{y_c = c_1 e^{5x} + c_2 x e^{5x}}$$

Particular Solution : $y_p = Ax + B$

$y_p' = A$

$y_p'' = 0$

$$y_p'' - 10y_p' + 25y_p = -10A + 25Ax + 25B$$

$$= 25Ax + (25B - 10A) = 30x + 3$$

$$\begin{cases} 25A = 30 \\ 25B - 10A = 3 \end{cases}$$

$A = 6/5$

$B = \frac{1}{25}(10A + 3) = \frac{1}{25} \cdot 15$ $B = 3/5$

$$\boxed{y_p = \frac{6}{5}x + \frac{3}{5}}$$

General Solution : $\boxed{y = c_1 e^{5x} + c_2 x e^{5x} + \frac{6}{5}x + \frac{3}{5}}$

5 $y'' - y' = -3$

Complementary Solution : $y'' - y' = 0$

$m^2 - m = 0 \Rightarrow m(m-1) = 0 \Rightarrow m = 0, 1$

$$\boxed{y_c = c_1 + c_2 e^x}$$

Particular Solution : $y_p = Ax$ (the initial guess $y_p = A$ is contained in y_c!)

$y_p' = A$; $y_p'' = 0$

$$\Rightarrow \left.\begin{aligned} y_p'' - y_p' &= -A \\ &= -3 \end{aligned}\right\} \Rightarrow A = 3 \Rightarrow \boxed{y_p = 3x}$$

General Solution : $\boxed{y = c_1 + c_2 e^x + 3x}$

⑥ $\frac{1}{4}y'' + y' + y = x^2 - 2x$

Complementary Solution: $\frac{1}{4}y'' + y' + y = 0$

$$\frac{1}{4}m^2 + m + 1 = 0$$
$$m^2 + 4m + 4 = 0$$
$$(m+2)^2 = 0$$

$$\boxed{y_c = c_1 e^{-2x} + c_2 x e^{-2x}}$$

Particular Solution: $y_p = Ax^2 + Bx + C$

$$y_p' = 2Ax + B$$
$$y_p'' = 2A$$

$$\frac{1}{4}y_p'' + y_p' + y_p = \frac{1}{2}A + 2Ax + B + Ax^2 + Bx + C$$
$$= Ax^2 + (2A+B)x + \left(\frac{1}{2}A + B + C\right) = x^2 - 2x$$

$$\Rightarrow \begin{cases} A = 1 \\ 2A + B = -2 \Rightarrow 2 + B = -2 \Rightarrow B = -4 \\ \frac{1}{2}A + B + C = 0 \Rightarrow \frac{1}{2} - 4 + C = 0 \Rightarrow C = 7/2 \end{cases}$$

$$\boxed{y_p = x^2 - 4x + 7/2}$$

General Solution: $\boxed{y = c_1 e^{-2x} + c_2 x e^{-2x} + x^2 - 4x + 7/2}$

⑦ $y'' + 3y = -48x^2 e^{3x}$

Complementary Solution: $y'' + 3y = 0$

$$m^2 + 3 = 0 \Rightarrow m = \pm i\sqrt{3}$$

$$\boxed{y_c = c_1 \sin(\sqrt{3}x) + c_2 \cos(\sqrt{3}x)}$$

Particular Solution: $y_p = (Ax^2 + Bx + C)e^{3x}$

$$y_p' = (2Ax + B)e^{3x} + 3(Ax^2 + Bx + C)e^{3x}$$
$$= (3Ax^2 + (3B + 2A)x + (B + 3C))e^{3x}$$
$$y_p'' = (6Ax + (3B + 2A) + 9Ax^2 + (9B + 6A)x + (3B + 9C))e^{3x}$$
$$= (9Ax^2 + (9B + 12A)x + (6B + 2A + 9C))e^{3x}$$

$$y_p'' + 3y_p = (12Ax^2 + (12B + 12A)x + (6B + 2A + 12C))e^{3x} = -48x^2 e^{3x}$$

$$\Rightarrow \begin{cases} 12A = -48 \\ 12B + 12A = 0 \\ 6B + 2A + 12C = 0 \end{cases} \Rightarrow \begin{cases} A = -4 \\ B = -A \Rightarrow B = 4 \\ 12C = -24 + 8 = -16 \Rightarrow C = -4/3 \end{cases}$$

$$\boxed{y_p = (-4x^2 + 4x - 4/3)e^{3x}}$$

General Solution: $\boxed{y = c_1 \sin(\sqrt{3}x) + c_2 \cos(\sqrt{3}x) - (4x^2 - 4x + 4/3)e^{3x}}$

⑧ $y'' + y = 2x\sin x$

Complementary Solution : $y''+y=0$

$m^2+1=0$

$m=\pm i$

$$\boxed{y_c = C_1 \sin x + C_2 \cos x}$$

Particular Solution : Initial guess would be $y_p = (Ax+B)\sin x + (Cx+D)\cos x$ but this duplicates the complementary solution. So, we multiply by x (and duplication is eliminated):

$$y_p = (Ax^2+Bx)\sin x + (Cx^2+Dx)\cos x$$

$$y_p' = (2Ax+B)\sin x + (Ax^2+Bx)\cos x - (Cx^2+Dx)\sin x + (2Cx+D)\cos x$$

$$= \left(-Cx^2+(2A-D)x+B\right)\sin x + \left(Ax^2+(B+2C)x+D\right)\cos x$$

$$y_p'' = (-Cx^2+(2A-D)x+B)\cos x + (-2Cx+(2A-D))\sin x + (2Ax+(B+2C))\cos x - (Ax^2+(B+2C)x+D)\sin x$$

$$= \left(-Cx^2+(4A-D)x+(2B+2C)\right)\cos x + \left(-Ax^2-(B+4C)x+(2A-2D)\right)\sin x$$

$$\Rightarrow y_p''+y_p = \left(-4Cx+(2A-2D)\right)\sin x + \left(4Ax+(2B+2C)\right)\cos x$$

$$= 2x\sin x$$

$$\Rightarrow \begin{cases} -4C=2 \\ 2A-2D=0 \\ 4A=0 \\ 2B+2C=0 \end{cases} \qquad \begin{cases} C=-1/2 \\ A=0 \\ D=0 \\ B=1/2 \end{cases}$$

$$\boxed{y_p = \tfrac{1}{2}x\sin x - \tfrac{1}{2}x^2\cos x}$$

General Solution : $\boxed{y = C_1\sin x + C_2\cos x + \tfrac{1}{2}x\sin x - \tfrac{1}{2}x^2\cos x}$

⑨ $y'' - y' + \frac{1}{4}y = 3 + e^{x/2}$

Complementary Solution : $y'' - y' + \frac{1}{4}y = 0$

$m^2 - m + \frac{1}{4} = 0$

$(m - 1/2)^2 = 0$

$$\boxed{y_c = c_1 e^{x/2} + c_2 x e^{x/2}}$$

Particular Solution : Initial guess would be $A + \underbrace{Be^{x/2}}$

↳ this part reproduces the y_c, so multiply by $\underline{\underline{x^2}}$ (eliminates duplication)

$y_p = A + Bx^2 e^{x/2}$

Q: Why didn't we also multiply the A by x^2?

In problem ⑧ we had $g(x) = 2x\sin x$ and so we multiplied the initial guess $[(Ax+B)\sin x + (Cx+D)\cos x]$ by x ~~> both terms were multiplied.

So why not here? Because here A and $Be^{x/2}$ are actually 2 different guesses for two particular solutions – we are really doing superposition here!

#9 : $g(x) = 3 + e^{x/2}$ (sum of functions)

guess : A

guess : $Be^{x/2}$ | $\cdot x^2$ to avoid y_c

#8 : $g(x) = 2x\sin x$ (one function)

guess : $[(Ax+B)\sin x + (Cx+D)\cos x] \cdot x$

this does have two terms, but they both come from $g(x)$.

If in #8 $g(x)$ were instead $g(x) = 2x\sin x + e^{3x}$ we would leave the guess for e^{3x} alone, i.e. we would put $y_p = (Ax^2+Bx)\sin x + (Cx^2+Dx)\cos x + Ee^{3x}$

$y_p' = 2Bxe^{x/2} + \frac{1}{2}Bx^2e^{x/2}$

$y_p'' = 2Be^{x/2} + Bxe^{x/2} + Bxe^{x/2} + \frac{1}{4}Bx^2e^{x/2}$

$y_p'' - y_p' + \frac{1}{4}y_p =$

$\left(2B + 2Bx + \frac{1}{4}Bx^2\right)e^{x/2}$

$- \left(2Bx + \frac{1}{2}Bx^2\right)e^{x/2}$

$+ \frac{1}{4}A + \frac{1}{4}Bx^2e^{x/2}$

$= \frac{1}{4}A + 2Be^{x/2}$

$= 3 + e^{x/2}$

$\Rightarrow \begin{cases} \frac{1}{4}A = 3 \\ 2B = 1 \end{cases} \quad \begin{cases} A = 12 \\ B = 1/2 \end{cases}$

$$\boxed{y_p = 12 + \frac{1}{2}x^2e^{x/2}}$$

General Solution : $\boxed{y = c_1e^{x/2} + c_2xe^{x/2} + \frac{1}{2}x^2e^{x/2} + 12}$

(10) $y'' - 2y' + 5y = e^x\cos(2x)$

Complementary Solution : $y'' - 2y' + 5y = 0$

$$m^2 - 2m + 5 = 0$$

$$\Delta = 4 - 20 = -16 \Rightarrow m = \frac{2 \pm 4i}{2} = 1 \pm 2i$$

$$\boxed{y_c = e^x(c_1 \sin(2x) + c_2\cos(2x))}$$

Particular Solution : Initial guess would be : $Ae^x\cos(2x) + Be^x\sin(2x)$
(multiply by x to eliminate duplication of y_c) :

$$y_p = Axe^x\cos(2x) + Bxe^x\sin(2x)$$

$$y_p' = Ae^x\cos(2x) + Axe^x\cos(2x) - 2Axe^x\sin(2x) + Be^x\sin(2x) + Bxe^x\sin(2x) + 2Bxe^x\cos(2x)$$

$$= \left[\left(A + (A+2B)x\right)\cos(2x) + \left(B + (B-2A)x\right)\sin(2x)\right]e^x$$

$$y_p'' = \left[\left(A + (A+2B)x\right)\cos(2x) + \left(B + (B-2A)x\right)\sin(2x) + (A+2B)\cos(2x) - 2\left(A + (A+2B)x\right)\sin(2x) + (B-2A)\sin(2x) + 2\left(B + (B-2A)x\right)\cos(2x)\right]e^x$$

$$= \left[\left(2A + 4B + (4B-3A)x\right)\cos(2x) + \left(2B - 4A + (-4A-3B)x\right)\sin(2x)\right]e^x$$

$$y_p'' - 2y_p' + 5y_p = (4B)\cos(2x)e^x + (-4A)\sin(2x)e^x$$
$$= e^x\cos(2x)$$

$$\Rightarrow \begin{cases} 4B = 1 \\ -4A = 0 \end{cases} \quad \begin{cases} B = 1/4 \\ A = 0 \end{cases} \qquad \boxed{y_p = \frac{1}{4}xe^x\sin(2x)}$$

General Solution : $\boxed{y = c_1 e^x\sin(2x) + c_2 e^x\cos(2x) + \frac{1}{4}xe^x\sin(2x)}$

11 $y''' - 6y'' = 3 - \cos x$

Complementary Solution :

$$y''' - 6y'' = 0$$
$$m^3 - 6m^2 = 0$$
$$m^2(m-6) = 0$$

$$\boxed{y_c = C_1 + C_2 x + C_3 e^{6x}}$$

Particular Solution :

$$y_p = \underbrace{Ax^2}_{} + \underbrace{B\cos x + C\sin x}_{\text{these come from } \cos x}$$

this comes from the constant 3. Initial guess would be just "A" but we must multiply by x^2 to eliminate duplication of the C_1 and $C_2 x$ in y_c.

$$y_p = Ax^2 + B\cos x + C\sin x$$
$$y_p' = 2Ax - B\sin x + C\cos x$$
$$y_p'' = 2A - B\cos x - C\sin x$$
$$y_p''' = B\sin x - C\cos x$$

$$y_p''' - 6y_p'' = -12A + (6B - C)\cos x + (6C + B)\sin x$$
$$= 3 - \cos x$$

$$\begin{cases} -12A = 3 \\ 6B - C = -1 \\ 6C + B = 0 \end{cases}$$

$A = -1/4$

$C = 6B + 1$

$36B + 6 + B = 0$ $\quad B = -6/37$

$\Rightarrow C = 1/37$

$$\boxed{y_p = -\frac{1}{4}x^2 - \frac{6}{37}\cos x + \frac{1}{37}\sin x}$$

General Solution :

$$\boxed{y = C_1 + C_2 x + C_3 e^{6x} - \frac{1}{4}x^2 - \frac{6}{37}\cos x + \frac{1}{37}\sin x}$$

12) $y'' + y = 8\sin^2 x$

Complementary Solution : $\boxed{y_c = C_1 \sin x + C_2 \cos x}$

Particular Solution : How to handle $8\sin^2 x$? Trigonometric Identity :

$$\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha.$$

$$\Rightarrow \boxed{\sin^2\alpha = \frac{1-\cos(2\alpha)}{2}}$$

$$y'' + y = 4 - 4\cos(2x)$$

$$y_p = A + B\cos(2x) + C\sin(2x)$$

$$y_p' = -2B\sin(2x) + 2C\cos(2x)$$

$$y_p'' = -4B\cos(2x) - 4C\sin(2x)$$

$$y_p'' + y_p = A - 3B\cos(2x) - 3C\sin(2x) = 4 - 4\cos(2x)$$

$$\begin{cases} A = 4 \\ 3B = 4 \Rightarrow B = 4/3 \\ 3C = 0 \Rightarrow C = 0 \end{cases}$$

$$\boxed{y_p = 4 + \frac{4}{3}\cos(2x)}$$

General Solution : $\boxed{y = C_1 \sin x + C_2 \cos x + 4 + \frac{4}{3}\cos(2x)}$

13. $y'' + 4y = -2$; $y(\pi/8) = 1/2$; $y'(\pi/8) = 2$.

Complementary Solution : $y'' + 4y = 0$
$m^2 + 4 = 0$
$m = \pm 2i$

$$\boxed{y_c = C_1 \sin(2X) + C_2 \cos(2X)}$$

Particular Solution : $y_p = A \Rightarrow y_p' = y_p'' = 0$

$\Rightarrow y_p'' + 4y_p = 4A = -2 \Rightarrow A = -1/2$; $y_p = -1/2$

General Solution : $\boxed{y = C_1 \sin(2X) + C_2 \cos(2X) - 1/2}$

$y' = 2C_1 \cos(2X) - 2C_2 \sin(2X)$

IVP : $y(\pi/8) = C_1 \frac{\sqrt{2}}{2} + C_2 \frac{\sqrt{2}}{2} - 1/2 = 1/2 \Rightarrow (C_1 + C_2)\frac{1}{\sqrt{2}} = 1 \Rightarrow C_1 + C_2 = \sqrt{2}$

$y'(\pi/8) = 2C_1 \frac{\sqrt{2}}{2} - 2C_2 \frac{\sqrt{2}}{2} = 2 \Rightarrow (C_1 - C_2)\sqrt{2} = 2 \Rightarrow C_1 - C_2 = \sqrt{2}$

$2C_1 = 2\sqrt{2}$

$C_1 = \sqrt{2}$; $C_2 = 0$

$$\boxed{y = \sqrt{2} \sin(2X) - 1/2}$$

14. $5y'' + y' = -6X$; $y(0) = 0$; $y'(0) = -10$

Complementary Solution : $5m^2 + m = 0$
$m(5m + 1) = 0 \quad \{0, -1/5\}$

$$\boxed{y_c = C_1 + C_2 e^{-X/5}}$$

Particular Solution : $y_p = AX^2 + BX$ (initial guess $AX + B$ duplicates C_1 in y_c)

$y_p' = 2AX + B$; $y_p'' = 2A$

$5y_p'' + y_p' = 10A + 2AX + B = 2AX + (10A + B) = -6X$

$\begin{cases} 2A = -6 \\ 10A + B = 0 \end{cases}$ $A = -3$; $B = 30$

$$\boxed{y_p = -3X^2 + 30X}$$

General Solution : $\boxed{y = C_1 + C_2 e^{-X/5} - 3X^2 + 30X}$

$y' = -\frac{1}{5}C_2 e^{-X/5} - 6X + 30$

IVP : $0 = y(0) = C_1 + C_2$

$-10 = y'(0) = -\frac{1}{5}C_2 + 30$

$C_1 + C_2 = 0$

$-\frac{1}{5}C_2 = -40 \quad C_2 = 200 \quad C_1 = -200$

$$\boxed{y = -200 + 200e^{-X/5} - 3X^2 + 30X}$$

15 $y'' + y = x^2 + 1$; $y(0) = 5$, $y(1) = 0$

Complementary Solution : $\boxed{y_c = c_1 \sin x + c_2 \cos x}$

Particular Solution : $y_p = Ax^2 + Bx + C$

$y_p' = 2Ax + B$

$y_p'' = 2A$

$$\begin{aligned} y_p'' + y_p &= 2A + Ax^2 + Bx + C \\ &= Ax^2 + Bx + (2A + C) \\ &= x^2 + 1 \end{aligned}$$

$$\begin{cases} A = 1 \\ B = 0 \\ 2A + C = 1 \Rightarrow C = -1 \end{cases}$$

$$\boxed{y_p = x^2 - 1}$$

General Solution : $y = c_1 \sin x + c_2 \cos x + x^2 - 1.$

IVP : $5 = y(0) = c_2 - 1 \Rightarrow c_2 = 6$

$0 = y(1) = c_1 \sin 1 + c_2 \cos 1 \Rightarrow c_1 \sin 1 = -6 \cos 1 \Rightarrow c_1 = -6\cot(1)$

$$\boxed{y = -6\cot(1) \sin x + 6 \cos x + x^2 - 1}$$