Math 2552 - Differential Equations **Homework 8**
Sections F1 – F4; L1 – L4
Georgia Institute of Technology, Fall 2015

Non-Homogeneous Linear ODEs with Constant Coefficients
Method of Undetermined Coefficients

Find the general solution for the following differential equations:

1. $y'' + 3y' + 2y = 6$.

2. $y'' + 4y = 3\sin(2x)$.

3. $y'' + 2y' + y = \sin x + 3\cos(2x)$.

4. $y'' - 10y' + 25y = 30x + 3$.

5. $y'' - y' = -3$.

6. $\frac{1}{4}y'' + y' + y = x^2 - 2x$.

7. $y'' + 3y = -48x^2e^{3x}$.

8. $y'' + y = 2x\sin x$.

9. $y'' - y' + \frac{1}{4}y = 3 + e^{x/2}$.

10. $y'' - 2y' + 5y = e^x\cos(2x)$.

11. $y''' - 6y'' = 3 - \cos x$.

12. $y'' + y = 8\sin^2 x$.

(You may need to use a certain double-angle trigonometric identity for number 12.)

A few IVPs and BVPs:

13. $y'' + 4y = -2;\quad y(\pi/8) = 1/2,\ y'(\pi/8) = 2$.

14. $5y'' + y' = -6x;\quad y(0) = 0,\ y'(0) = -10$.

15. $y'' + y = x^2 + 1;\quad y(0) = 5,\ y(1) = 0$.

Non-homogeneous Linear ODEs with Constant Coefficients

– Method of Undetermined Coefficients –

Method applies to equations of the form:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = g(x)$$

where:

- $a_n, a_{n-1}, \dots, a_1, a_0$ are constants
- $g(x)$ is:
  - a polynomial (includes constant functions)
  - exponential $e^{\alpha x}$
  - $\sin(\beta x)$ or $\cos(\beta x)$
  - sums and products of these

General Idea: This method is for finding a particular solution y_p.

- Make an assumption about the form of y_p (some type of function with unknown coefficients), then find the coefficients.
- General rule for the assumption: linear combination of all the linearly independent types of functions that arise from repeated differentiations of $g(x)$.
- Polynomials, exponentials, and sin & cos work because any sum or product of them gives a sum or product of these same types of functions through repeated differentiation.

Superposition Principle for Non-Homogeneous Linear ODEs:

Suppose that for every $j \in \{1, 2, \dots, k\}$, y_{p_j} is a particular solution for

$$a_n(x) y^{(n)} + \dots + a_1(x) y' + a_0(x) y = g_j(x).$$

Then $y_{p_1} + \dots + y_{p_k}$ is a particular solution for

$$a_n(x) y^{(n)} + \dots + a_1(x) y' + a_0(x) y = g_1(x) + \dots + g_k(x).$$

Case 1: No function in the assumed y_p duplicates any part of y_c (i.e. no function in the assumed y_p is a solution to the associated homogeneous equation).

$g(x)$	y_p
Polynomial $p(x)$ of degree $K \geq 0$	Polynomial of same degree K
$g(x) = x+1$	$y_p = Ax+B$
$g(x) = x^3 + x - 4$	$y_p = Ax^3 + Bx^2 + Cx + D$
$g(x) = 2$	$y_p = A$
Exponential $e^{\alpha x}$	$Ae^{\alpha x}$
$g(x) = e^{9x}$	$y_p = Ae^{9x}$
$\sin(\beta x)$ or $\cos(\beta x)$	$A\sin(\beta x) + B\cos(\beta x)$
$g(x) = \sin(2x)$	$y_p = A\sin(2x) + B\cos(2x)$
$g(x) = \cos(7x)$	$y_p = A\sin(7x) + B\cos(7x)$
$p(x)e^{\alpha x}$	(Same degree polynomial) $\cdot e^{\alpha x}$
$g(x) = (x+1)e^{3x}$	$y_p = (Ax+B)e^{3x}$
$p(x)\sin(\beta x)$ or $p(x)\cos(\beta x)$	(Same deg. poly) $\cdot \sin(\beta x)$ + (Same deg. poly) $\cdot \cos(\beta x)$
$g(x) = (x-1)\sin(2x)$	$y_p = (Ax+B)\sin(2x) + (Cx+D)\cos(2x)$
$g(x) = x^2\cos(3x)$	$y_p = (Ax^2+Bx+C)\sin(3x) + (Dx^2+Ex+F)\cos(3x)$
$e^{\alpha x}\sin(\beta x)$ or $e^{\alpha x}\cos(\beta x)$	$Ae^{\alpha x}\sin(\beta x) + Be^{\alpha x}\cos(\beta x)$
$g(x) = e^{2x}\sin(7x)$	$y_p = Ae^{2x}\sin(7x) + Be^{2x}\cos(7x)$
$p(x)e^{\alpha x}\sin(\beta x)$ or $p(x)e^{\alpha x}\cos(\beta x)$	(Same deg. poly.) $e^{\alpha x}\sin(\beta x)$ + (Same deg. poly.) $e^{\alpha x}\cos(\beta x)$
$g(x) = (x+1)e^{2x}\cos(3x)$	$y_p = (Ax+B)e^{2x}\sin(3x) + (Cx+D)e^{2x}\cos(3x)$
Sums of Functions of the Types Above	**Sum of Individual Guesses For y_p (Superposition)**
$g(x) = x^2 + e^{7x}\cos(\pi x)$	$y_p = Ax^2+Bx+C + De^{7x}\sin(\pi x) + Ee^{7x}\cos(\pi x)$
$g(x) = x\sin(2x) + e^{3x} + x^3$	$y_p = (Ax+B)\sin(2x) + (Cx+D)\cos(2x) + Ee^{3x} + Fx^3 + Gx^2 + Hx + I.$

Example 1: $y'' - 2y' - 3y = e^{2x} + 3x^2 + 4x - 5 + 5\cos(2x)$

Complementary Solution: $y'' - 2y' - 3y = 0$

$y_c = c_1 e^{-x} + c_2 e^{3x}$

$m^2 - 2m - 3 = 0 \Rightarrow (m-3)(m+1) = 0 \Rightarrow m \in \{-1, 3\}$

Particular Solution: $y_p = Ae^{2x} + Bx^2 + Cx + D + E\cos(2x) + F\sin(2x)$

$y_p' = 2Ae^{2x} + 2Bx + C - 2E\sin(2x) + 2F\cos(2x)$

$y_p'' = 4Ae^{2x} + 2B - 4E\cos(2x) - 4F\sin(2x)$

$$\Rightarrow y_p'' - 2y_p' - 3y_p = e^{2x}\underbrace{(4A - 4A - 3A)}_{1} + x^2\underbrace{(-3B)}_{3} + x\underbrace{(-4B - 3C)}_{4} + \underbrace{(2B - 2C - 3D)}_{-5}$$
$$+ \cos(2x)\underbrace{(-4E - 4F - 3E)}_{5} + \sin(2x)\underbrace{(-4F + 4E - 3F)}_{0}$$

$$\begin{cases} -3A = 1 \Rightarrow A = -1/3 \\ -3B = 3 \Rightarrow B = -1 \\ -4B - 3C = 4 \Rightarrow C = 0 \\ 2B - 2C - 3D = -5 \Rightarrow D = 1 \\ -7E - 4F = 5 \\ 4E - 7F = 0 \Rightarrow F = \frac{4}{7}E \end{cases}$$

$$\Rightarrow -7E + \frac{16}{7}E = -5 \Rightarrow \frac{65}{7}E = -5 \Rightarrow E = -\frac{7}{13} \Rightarrow F = -\frac{4}{13}$$

$$y_p = -\frac{1}{3}e^{2x} - x^2 + 1 - \frac{7}{13}\cos(2x) - \frac{4}{13}\sin(2x)$$

$\Rightarrow$ **General Solution**: $\boxed{y = c_1 e^{-x} + c_2 e^{3x} - \frac{1}{3}e^{2x} - x^2 + 1 - \frac{7}{13}\cos(2x) - \frac{4}{13}\sin(2x)}$

Remark: with superposition problems like this, you can also solve each non-homogeneous equation individually and add the results at the end. For example:

$$\left.\begin{aligned} y'' - 2y' - 3y = e^{2x} &\Rightarrow y_{p_1} = -\frac{1}{3}e^{2x} \\ y'' - 2y' - 3y = 3x^2 + 4x - 5 &\Rightarrow y_{p_2} = -x^2 + 1 \\ y'' - 2y' - 3y = 5\cos(2x) &\Rightarrow y_{p_3} = -\frac{7}{13}\cos(2x) - \frac{4}{13}\sin(2x) \end{aligned}\right\} \Rightarrow y_p = y_{p_1} + y_{p_2} + y_{p_3}.$$

Case (2): Suppose $g(x) = g_1(x) + \dots + g_k(x)$, where each function $g_j(x)$ is of the type previously discussed, with corresponding guess y_{p_j} for its particular solution. If any y_{p_j} contains a function in y_c, then multiply this y_{p_j} by the lowest power (x^n) of x that eliminates the duplication.

Example (2): $\boxed{y'' - y' - 2y = g(x)}$

Complementary Solution: $y_c = c_1 e^{2x} + c_2 e^{-x}$

$m^2 - m - 2 = 0 \Rightarrow (m-2)(m+1) = 0 \Rightarrow m_1 = 2;\ m_2 = -1$

$g(x)$	y_p
$g(x) = e^{3x}$	$y_p = Ae^{3x}$ (no duplication)
$g(x) = e^{2x}$	$y_p = Axe^{2x}$ Original guess: Ae^{2x} duplicates part of y_c Multiplication by x eliminates duplication
$g(x) = 3e^{2x} + \cos(7x)$	$y_p = Axe^{2x} + B\cos(7x) + C\sin(7x)$ Superposition: $g_1(x) = 3e^{2x}$; as above, multiply by x $g_2(x) = \cos(7x)$; original guess $y_{p_2} = B\cos(7x) + C\sin(7x)$ does not duplicate any part of y_c. Careful: $y_p = x\left[Ae^{2x} + B\cos(7x) + C\sin(7x)\right]$ is wrong!
$g(x) = xe^{2x}$	$y_p = (Ax^2 + Bx)e^{2x}$ Original guess: $(Ax+B)e^{2x} = Axe^{2x} + Be^{2x}$ (Be^{2x} duplicates y_c) Multiply original guess by x (removes duplication)
$g(x) = \sin(3x)e^{2x}$	$y_p = A\sin(3x)e^{2x} + B\cos(3x)e^{2x}$ (no duplication)
$g(x) = x^2e^{2x} + e^{\pi x}$	$y_p = (Ax^3 + Bx^2 + Cx)e^{2x} + De^{\pi x}$ (Superposition + Duplication)

Example (3): $y'' + 16 = g(x)$

Complementary Solution: $y_c = C_1 \cos(4x) + C_2 \sin(4x)$

$m^2 + 16 = 0$; $m = \pm 4i$

$g(x)$	y_p
$g(x) = x^2 e^x$	$y_p = (Ax^2 + Bx + C)e^x$ (no duplication)
$g(x) = e^x \sin(4x)$	$y_p = Ae^x \sin(4x) + Be^x \cos(4x)$ (no duplication)
$g(x) = 2\cos(4x)$	$y_p = Ax\cos(4x) + Bx\sin(4x)$ Original guess $(A\cos(4x) + B\sin(4x))$ duplicates part of y_c.
$g(x) = x^3\cos(4x) + e^{2x}\sin(4x)$	$y_p = (Ax^4 + Bx^3 + Cx^2 + Dx)\cos(4x) + (Ex^4 + Fx^3 + Gx^2 + Hx)\sin(4x) + [Ie^{2x}\sin(4x) + Je^{2x}\cos(4x)]$ Superposition: $g_1(x) = x^3\cos(4x)$, original guess: $(Ax^3 + Bx^2 + Cx + D)\cos(4x) + (Ex^3 + Fx^2 + Gx + H)\sin(4x)$ duplicates part of y_c because of the terms $D\cos(4x)$ and $H\sin(4x)$ $\Rightarrow$ multiply by x, duplication eliminated. $g_2(x) = e^{2x}\sin(4x) \Rightarrow y_{p_2} = Ie^{2x}\sin(4x) + Je^{2x}\cos(4x)$ (no duplication)

Example (4): $y'' - 2y' + y = 2e^x + x$

Complementary Solution: $y_c = C_1 e^x + C_2 x e^x$

Particular Solution: $y_p = Ax^2e^x + (Bx + C)$

$g_2(x) = x$; $y_{p_2} = Bx + C$ ✓

$g_1(x) = 2e^x$; $y_{p_1} = Ae^x \cdot 1 \cdot x^2$

duplicates y_c; $xy_{p_1} = Axe^x$ also does