Homework (7)
– Solutions –

Homogeneous Linear ODEs with Constant Coefficients

(1) $4y''+y'=0$

Char. Eq.: $4m^2+m=0$

$m(4m+1)=0$

$m=0,-1/4$

$$y=c_1+c_2e^{-x/4}$$

(2) $y''-36y=0$

Char. Eq.: $m^2-36=0$

$(m-6)(m+6)=0$

$m=6,-6$

$$y=c_1e^{6x}+c_2e^{-6x}$$

(3) $12y''-5y'-2y=0$

Char. Eq.: $12m^2-5m-2=0$

$\Delta=25+96=121$

$m=\frac{5\pm11}{24}=\frac{2}{3};-\frac{1}{4}$

$$y=c_1e^{2/3x}+c_2e^{-1/4x}$$

(4) $y''-4y'+5y=0$

Char. Eq.: $m^2-4m+5=0$

$\Delta=16-20=-4$

$m=\frac{4\pm2i}{2}=2\pm i$

$$y=e^{2x}(c_1\sin x+c_2\cos x)$$

(5) $3y''+2y'+y=0$

Char. Eq.: $3m^2+2m+1=0$

$\Delta=4-12=-8$

$m=\frac{-2\pm2\sqrt{2}i}{6}$

$=-\frac{1}{3}\pm\frac{\sqrt{2}}{3}i$

$$y=e^{-x/3}\left(c_1\sin\left(\tfrac{\sqrt{2}}{3}x\right)+c_2\cos\left(\tfrac{\sqrt{2}}{3}x\right)\right)$$

(6) $y''' + y'' - 2y = 0$

Char. Eq.: $m^3 + m^2 - 2 = 0$

$m^3 - 1 + m^2 - 1 = 0$

$(m-1)(m^2+m+1) + (m-1)(m+1) = 0$

$(m-1)(m^2+2m+2) = 0$

$m = 1$

$\Delta = 4 - 8 = -4$

$\frac{-2 \pm 2i}{2}$ $\quad -1 \pm i$

$$y = c_1 e^x + e^{-x}(c_2 \sin x + c_3 \cos x)$$

(7) $y^{(4)} + y^{(3)} + y'' = 0$

Char. Eq.: $m^4 + m^3 + m^2 = 0$

$m^2(m^2+m+1) = 0$

$m = 0$ (x2) $\quad -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$

$$y = c_1 + c_2 x + e^{-x/2}\left(c_3 \sin\left(\frac{\sqrt{3}}{2}x\right) + c_4 \cos\left(\frac{\sqrt{3}}{2}x\right)\right)$$

(8) $y''' + 3y'' + 3y' + y = 0$

Char. Eq.: $m^3 + 3m^2 + 3m + 1 = 0$

$(m+1)^3 = 0$

$m = -1$ (x3)

$$y = c_1 e^{-x} + c_2 x e^{-x} + c_3 x^2 e^{-x}$$

(9) $y^{(5)} - 16y' = 0$

Char. Eq.: $m^5 - 16m = 0$

$m(m^4 - 16) = 0$

$m(m^2-4)(m^2+4) = 0$

$m(m-2)(m+2)(m^2+4) = 0$

$0 \quad 2 \quad -2 \quad \pm 2i$

$$y = c_1 + c_2 e^{2x} + c_3 e^{-2x} + c_4 \sin(2x) + c_5 \cos(2x)$$

(10) $y^{(5)}+5y^{(4)}-2y'''-10y''+y'+5y=0$

Char. Eq.: $m^5+5m^4-2m^3-10m^2+m+5=0$

$(m+5)(m^4-2m^2+1)=0$

$(m+5)(m^2-1)^2=0$

$(m+5)(m-1)^2(m+1)^2=0$

-5 ; 1×2 ; -1×2

$$y=c_1e^{-5x}+c_2e^{x}+c_3xe^{x}+c_4e^{-x}+c_5xe^{-x}$$

(11) $y^{(4)}-7y''-18y=0$

Char. Eq.: $m^4-7m^2-18=0$

$u=m^2$

$u^2-7u-18=0$

$(u-9)(u+2)=0$

$u=9$; $u=-2$

$m^2=9$; $m^2=-2$

3 ; -3 ; $\pm i\sqrt{2}$

$$y=c_1e^{3x}+c_2e^{-3x}+c_3\sin(\sqrt{2}x)+c_4\cos(\sqrt{2}x)$$

(12) $y^{(5)}-2y^{(4)}+17y'''=0$

Char. Eq.: $m^5-2m^4+17m^3=0$

$m^3(m^2-2m+17)=0$

0×3

$\Delta=4-4\cdot 17=-4\cdot 16$

$m=\frac{2\pm 8i}{2}$

$=1\pm 4i$

$$y=c_1+c_2x+c_3x^2+e^{x}\left(c_4\sin(4x)+c_5\cos(4x)\right)$$

(13) $y'' + 16y = 0$; $y(0) = 2$, $y'(0) = -2$.

Char. Eq.: $m^2 + 16 = 0$

$m = \pm 4i$

$$\boxed{y = c_1 \sin(4x) + c_2 \cos(4x)}$$

$2 = y(0) = c_2 \Rightarrow \boxed{c_2 = 2}$

$\Rightarrow y' = 4c_1 \cos(4x) - 4c_2 \sin(4x)$

$-2 = y'(0) = 4c_1 \Rightarrow \boxed{c_1 = -1/2}$

$$\boxed{y = -\frac{1}{2}\sin(4x) + 2\cos(4x)}$$

(14) $y'' - y = 0$; $y(0) = y'(0) = 1$.

Char. Eq.: $m^2 - 1 = 0$

$(m-1)(m+1) = 0$

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$$\Rightarrow \boxed{y = c_1 e^x + c_2 e^{-x}} \quad ; 1 = y(0) = c_1 + c_2$$

$$y' = c_1 e^x - c_2 e^{-x} \quad ; 1 = y'(0) = c_1 - c_2$$

$$\begin{cases} c_1 + c_2 = 1 \\ c_1 - c_2 = 1 \end{cases}$$

$\oplus \quad 2c_1 = 2$

$\boxed{c_1 = 1} \Rightarrow \boxed{c_2 = 0}$

$$\boxed{y = e^x}$$

(15) $y'' + y = 0$; $y(\pi/3) = 0$; $y'(\pi/3) = 2$

$m^2 + 1 = 0$

$m = \boxed{\pm i}$

$y = c_1 \sin x + c_2 \cos x \Rightarrow y(\pi/3) = c_1 \cdot \frac{\sqrt{3}}{2} + c_2 \cdot \frac{1}{2} = 0$

$y' = c_1 \cos x - c_2 \sin x \Rightarrow y'(\pi/3) = c_1 \cdot \frac{1}{2} - c_2 \cdot \frac{\sqrt{3}}{2} = 2$

$$\begin{cases} c_1\sqrt{3} + c_2 = 0 \\ c_1 - c_2\sqrt{3} = 4 \end{cases} \qquad \begin{array}{l} c_1\sqrt{3} + c_2 = 0 \\ c_1\sqrt{3} - 3c_2 = 4\sqrt{3} \end{array}$$

$\ominus \quad 4c_2 = -4\sqrt{3}$

$\boxed{c_2 = -\sqrt{3}} \Rightarrow \boxed{c_1 = 1}$

$$\boxed{y = \sin x - \sqrt{3}\cos x}$$

(16) $y''' + 12y'' + 36y' = 0$; $y(0)=0$, $y'(0)=1$, $y''(0)=-7$.

$m^3 + 12m^2 + 36m = 0$

$m(m^2+12m+36) = 0$

$m(m+6)^2 = 0$

$m = 0$, $m = -6 \times 2$

$$y = c_1 + c_2 e^{-6x} + c_3 x e^{-6x}$$

$$y' = -6c_2 e^{-6x} + c_3 e^{-6x} - 6c_3 x e^{-6x}$$

$$y'' = +36c_2 e^{-6x} - 6c_3 e^{-6x} - 6c_3 e^{-6x} + 36c_3 x e^{-6x}$$

$$\begin{cases} 0 = y(0) = c_1 + c_2 \\ 1 = y'(0) = -6c_2 + c_3 \\ -7 = y''(0) = 36c_2 - 6c_3 - 6c_3 \end{cases} \qquad \begin{cases} c_1 + c_2 = 0 \\ -6c_2 + c_3 = 1 \\ 36c_2 - 12c_3 = -7 \end{cases} \longrightarrow c_1 = -c_2$$

$$-6c_2 + c_3 = 1$$

$$3c_2 - c_3 = -\frac{7}{12}$$

$$\oplus \quad -3c_2 = \frac{5}{12} \Rightarrow c_2 = -\frac{5}{36} \Rightarrow c_1 = \frac{5}{36}$$

$$c_3 = 1 + 6c_2 = 1 - \frac{5}{6} = \frac{1}{6} \qquad c_3 = 1/6$$

$$\boxed{y = \frac{5}{36} - \frac{5}{36} e^{-6x} + \frac{1}{6} x e^{-6x}}$$

(17) $y'' - 10y' + 25y = 0$; $y(0) = 1$, $y(1) = 0$

$m^2 - 10m + 25 = 0$

$(m-5)^2 = 0$

(5) ×2

$$y = c_1 e^{5x} + c_2 x e^{5x}$$

$y' = 5c_1 e^{5x} + c_2 e^{5x} + 5c_2 x e^{5x}$ ⇝ nm (not needed)

$1 = y(0) = c_1$ $\quad (c_1 = 1)$

$0 = y(1) = c_1 e^5 + c_2 e^5 \Rightarrow \quad c_2 e^5 = -e^5 \Rightarrow (c_2 = -1)$

$$\boxed{y = e^{5x} - x e^{5x}}$$

(18) $y'' + y = 0$; $y'(0) = 0$, $y'(\pi/2) = 2$

$m^2 + 1 = 0$

$m = \pm i$

$$y = c_1 \cos x + c_2 \sin x$$

$y' = -c_1 \sin x + c_2 \cos x$

$0 = y'(0) = c_2 \Rightarrow (c_2 = 0)$

$2 = y'(\pi/2) = -c_1 \Rightarrow (c_1 = -2)$

$$\boxed{y = -2\cos x}$$

(19) $y'' - y = 0$; $y(0) = 1$, $y'(1) = 0$

$m^2 - 1 = 0$

$(m-1)(m+1) = 0$

(1) (−1)

$y = c_1 e^x + c_2 e^{-x}$

$y' = c_1 e^x - c_2 e^{-x}$

$1 = y(0) = c_1 + c_2$

$0 = y'(1) = c_1 e - c_2 e^{-1}$

$$\begin{cases} c_1 + c_2 = 1 \\ c_1 - c_2 = 0 \end{cases}$$

c_2=e^2/(1+e^2)

c_1=1/(1+e^2)

(20) $y'' + 4y = 0$; $y(0) = 0$, $y(\pi) = 0$.

$m^2 + 4 = 0$ $\quad (\pm 2i)$

$y = c_1 \sin(2x) + c_2 \cos(2x)$

$0 = y(0) = c_2 \Rightarrow (c_2 = 0)$

$0 = y(\pi) = c_2 \Rightarrow (c_2 = 0)$ > c_1 can be anything !

$$\boxed{y = c \sin(2x)}$$ (∞-many solutions)

(21) Roots of char. eqn.: $m_1 = 4$; $m_2 = m_3 = -3$.

$\Rightarrow$ Char. eqn.: $(m-4)(m+3)^2 = 0$

$(m-4)(m^2+6m+9) = 0$

$m^3+6m^2+9m-4m^2-24m-36 = 0$

$m^3+2m^2-15m-36 = 0$

$$\boxed{y'''+2y''-15y'-36y = 0}$$

(22) Roots of char. eqn.: $m_1 = 1$; $m_2 = 2+i$; $m_3 = 2-i$.

$\Rightarrow$ Char. eqn.: $(m-1)(m-2-i)(m-2+i) = 0$

$(m-1)\left((m-2)^2+1\right)$

$(m-1)(m^2-4m+5)$

m^3-4m^2+5m
$\quad -m^2+4m-5$

m^3-5m^2+9m-5

$$\boxed{y'''-5y''+9y'-5y = 0}$$

(23) $4e^{6x}$, πe^{-2x}

these are coefficients, don't really matter here.

What does matter: the roots of the char. eqn. are 6 and -2

$(m-6)(m+2) = m^2-4m-12$

$$\boxed{y''-4y'-12y = 0}$$

(24) $100\cos(4x)$, $-20\sin(4x)$

Roots of char. eqn.: $\pm 4i$

$m^2+16 = 0$

$$\boxed{y''+16y = 0}$$