Math 2552 - Differential Equations **Homework 7**
Sections F1 – F4; L1 – L4
Georgia Institute of Technology, Fall 2015

Homogeneous Linear ODEs with Constant Coefficients

Find the general solution for the following differential equations:

1. $4y'' + y' = 0.$

2. $y'' - 36y = 0.$

3. $12y'' - 5y' - 2y = 0.$

4. $y'' - 4y' + 5y = 0.$

5. $3y'' + 2y' + y = 0.$

6. $y''' + y'' - 2y = 0.$

7. $y^{(4)} + y^{(3)} + y'' = 0.$

8. $y''' + 3y'' + 3y' + y = 0.$

9. $y^{(5)} - 16y' = 0.$

10. $y^{(5)} + 5y^{(4)} - 2y''' - 10y'' + y' + 5y = 0.$

11. $y^{(4)} - 7y'' - 18y = 0.$

12. $y^{(5)} - 2y^{(4)} + 17y''' = 0.$

Solve the following initial value problems:

13. $y'' + 16y = 0;\quad y(0) = 2,\ y'(0) = -2.$

14. $y'' - y = 0;\quad y(0) = y'(0) = 1.$

15. $y'' + y = 0;\quad y(\pi/3) = 0,\ y'(\pi/3) = 2.$

16. $y''' + 12y'' + 36y' = 0;\quad y(0) = 0,\ y'(0) = 1,\ y''(0) = -7.$

Solve the following boundary value problems:

17. $y'' - 10y' + 25y = 0;\quad y(0) = 1,\ y(1) = 0.$

18. $y'' + y = 0;\quad y'(0) = 0,\ y'(\pi/2) = 2.$

19. $y'' - y = 0;\quad y(0) = 1,\ y'(1) = 0.$

20. $y'' + 4y = 0;\quad y(0) = 0,\ y(\pi) = 0.$

Find the homogeneous linear ODE with constant coefficients whose characteristic equation has the roots:

21. $m_1 = 4;\ m_2 = m_3 = -3.$

22. $m_1 = 1;\ m_2 = 2 + i;\ m_3 = 2 - i.$

Find a homogeneous linear ODE with constant coefficients which could have the solution:

23. $4e^{6x} + \pi e^{-2x}.$

24. $100\cos(4x) - 20\sin(4x).$

Homogeneous Linear ODEs with Constant Coefficients

$$\boxed{\begin{gathered} a_n y^{(n)} + \ldots + a_1 y' + a_0 y = 0 \\ a_n, a_{n-1}, \ldots, a_1, a_0 \in \mathbb{R} \end{gathered}}$$

$\boxed{n=2}$: $ay'' + by' + c = 0$

Characteristic Equation : $\boxed{am^2 + bm + c = 0}$

- 2 distinct real roots m_1, m_2 : $y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$
- 1 repeated real root m_1 : $y = c_1 e^{m_1 x} + c_2 \underline{\underline{x}} e^{m_1 x}$
- 2 complex roots $m_{1,2} = \alpha \pm i\beta$: $y = e^{\alpha x}\left(c_1 \cos(\beta x) + c_2 \sin(\beta x)\right)$

$\boxed{\text{General } n}$: $a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = 0$

Characteristic Equation : $\boxed{a_n m^n + a_{n-1} m^{n-1} + \ldots + a_1 m + a_0 = 0}$

- For any real root m_1 of multiplicity $\underline{\underline{1}}$, the general solution must contain $c_1 e^{m_1 x}$
- For any real root m_1 of multiplicity $\underline{\underline{k}}$, the general solution must contain $c_1 e^{m_1 x} + c_2 x e^{m_1 x} + \ldots + c_k x^{k-1} e^{m_1 x}$
- For any pair of complex roots $\alpha \pm i\beta$, the general solution must contain $e^{\alpha x}\left(c_1 \cos(\beta x) + c_2 \sin(\beta x)\right)$
- If a complex root $\alpha + i\beta$ has multiplicity $\underline{\underline{k}}$, then so does its conjugate $\alpha - i\beta$, and the general solution must contain <u>linear combinations</u> of

$$\left\{ \begin{array}{c} e^{\alpha x}\cos(\beta x),\ e^{\alpha x}\sin(\beta x) \\ x e^{\alpha x}\cos(\beta x),\ x e^{\alpha x}\sin(\beta x) \\ \vdots \\ x^{k-1} e^{\alpha x}\cos(\beta x),\ x^{k-1} e^{\alpha x}\sin(\beta x) \end{array} \right\}$$

Examples :

- $y'' - 2y' + y = 0$

$m^2 - 2m + 1 = 0$

$(m-1)^2 = 0 \Rightarrow m = 1$ (repeated root) $\Rightarrow \boxed{y = c_1 e^x + c_2 x e^x}$

- $y'' + y' + y = 0$

$m^2 + m + 1 = 0$

$\Delta = 1 - 4 = -3 \Rightarrow m = \dfrac{-1 \pm i\sqrt{3}}{2} = \dfrac{-1}{2} \pm i\dfrac{\sqrt{3}}{2} \Rightarrow$

$$\boxed{y = e^{-x/2}\left(c_1 \cos\left(\tfrac{\sqrt{3}}{2}x\right) + c_2 \sin\left(\tfrac{\sqrt{3}}{2}x\right)\right)}$$

- $y'' + y' - 2y = 0$

$m^2 + m - 2 = 0$

$(m+2)(m-1) = 0$

$m_1 = -2;\ m_2 = 1$

$$\boxed{y = c_1 e^{-2x} + c_2 e^x}$$

Characteristic Equation :	General Solution :
$(m-2)(m^2+m+1)$ 2 $\quad -1/2 \pm \sqrt{3}/2\, i$	$c_1 e^{2x} + e^{-x/2}\left(c_2 \sin\left(\tfrac{\sqrt{3}}{2}x\right) + c_3 \cos\left(\tfrac{\sqrt{3}}{2}x\right)\right)$
$(m-2)^3 (m^2+16)$ 2×3 $\quad \pm 4i$	$c_1 e^{2x} + c_2 x e^{2x} + c_3 x^2 e^{2x}$ $+ c_4 \sin(4x) + c_5 \cos(4x)$
$(m+3)(m-2)^2(m-5)^3$ -3 $\quad 2 \times 2$ $\quad 5 \times 3$	$c_1 e^{-3x} + c_2 e^{2x} + c_3 x e^{2x}$ $+ c_4 e^{5x} + c_5 x e^{5x} + c_6 x^2 e^{5x}$
$(m^2+m+1)^2$ $-1/2 \pm \sqrt{3}/2\, i$ $\times 2$	$e^{-x/2}\left(c_1 \sin(\sqrt{3}/2\, x) + c_2 \cos(\sqrt{3}/2\, x)\right)$ $+ e^{-x/2} x \left(c_3 \sin(\sqrt{3}/2\, x) + c_4 \cos(\sqrt{3}/2\, x)\right).$