Higher-Order Linear ODEs

Homework ⑥
– Solutions –

1 $y = c_1 e^x \cos x + c_2 e^x \sin x$

$y'' - 2y' + 2y = 0$

$y' = c_1 e^x \cos x - c_1 e^x \sin x + c_2 e^x \sin x + c_2 e^x \cos x$
$= e^x \left((c_1 + c_2)\cos x + (c_2 - c_1)\sin x \right)$

(a). $y(0) = 1;\ y'(0) = 0$

$y(0) = 1 \Rightarrow c_1 = 1$

$y'(0) = 0 \Rightarrow c_1 + c_2 = 0 \Rightarrow c_2 = -1$

$\Rightarrow \boxed{y = e^x \cos x - e^x \sin x}$ one solution

(b). $y(0) = 1;\ y(\pi) = -1$

$y(0) = 1 \Rightarrow c_1 = 1$

$y(\pi) = -1 \Rightarrow -c_1 e^{\pi} = -1 \Rightarrow c_1 = e^{-\pi}$

contradiction! $\Rightarrow$ no solution

(c). $y(0) = 0;\ y(\pi) = 0$

$$\left.\begin{array}{l} y(0) = 0 \Rightarrow c_1 = 0 \\ y(\pi) = 0 \Rightarrow -c_1 e^{\pi} = 0 \Rightarrow c_1 = 0 \end{array}\right\} \Rightarrow \boxed{y = c_2 e^x \sin x,\ c_2 \in \mathbb{R}}$$

∞-many solutions

(d). $y(0) = 1;\ y(\pi/2) = 1$

$$\left.\begin{array}{l} y(0) = 1 \Rightarrow c_1 = 1 \\ y(\pi/2) = 1 \Rightarrow c_2 e^{\pi/2} = 1 \Rightarrow c_2 = e^{-\pi/2} \end{array}\right\} \Rightarrow \boxed{y = e^x \cos x + e^{-\pi/2} e^x \sin x}$$

one solution

2 (a). $f_1(x) = 5;\ f_2(x) = \cos^2 x;\ f_3(x) = \sin^2 x.$

$f_2(x) + f_3(x) = 1 = \frac{1}{5} f_1(x)$ for all x $\Rightarrow$ lin. dependent

(b). $f_1(x) = \cos(2x);\ f_2(x) = 1;\ f_3(x) = \cos^2 x$

$f_1(x) = 2\cos^2 x - 1 = 2 f_3(x) - f_2(x)$ for all x $\Rightarrow$ lin. dependent

(c). $f_1(x) = x;\ f_2(x) = x^2;\ f_3(x) = 4x - 3x^2$

$f_3(x) = 4 f_1(x) - 3 f_2(x)$ for all x $\Rightarrow$ lin. dependent

(d). $f_1(x) = x;\ f_2(x) = \sin x;\ f_3(x) = 3x;\ f_4(x) = e^x$

$f_3(x) = 3 f_1(x)$ for all x $\Rightarrow$ lin. dependent

$\Rightarrow \left(3 \cdot f_1(x) + 0 \cdot f_2(x) - 1 \cdot f_3(x) + 0 \cdot e^x = 0 \text{ for all } x. \right)$

3 (a) $\sqrt{x}, x^2$; $(0,\infty)$

$$W = \begin{vmatrix} \sqrt{x} & x^2 \\ \frac{1}{2\sqrt{x}} & 2x \end{vmatrix} = 2x\sqrt{x} - \frac{x^2}{2\sqrt{x}} = \frac{3}{2}x\sqrt{x} \neq 0 \text{ on } (0,\infty)$$

(b) $1+x, x^3$; $(-\infty,\infty)$.

$$W = \begin{vmatrix} 1+x & x^3 \\ 1 & 3x^2 \end{vmatrix} = 3x^2+3x^3-x^3 = 3x^2+2x^3 \neq 0 \text{ on } \mathbb{R}$$

(c) e^x, e^{-x}, e^{4x} ; $(-\infty,\infty)$

factor out an e^x from each row

R_2-R_1, R_3-R_1

$$W = \begin{vmatrix} e^x & e^{-x} & e^{4x} \\ e^x & -e^{-x} & 4e^{4x} \\ e^x & +e^{-x} & 16e^{4x} \end{vmatrix} = e^{3x}\begin{vmatrix} 1 & e^{-2x} & e^{3x} \\ 1 & -e^{-2x} & 4e^{3x} \\ 1 & e^{-2x} & 16e^{3x} \end{vmatrix} = e^{3x}\begin{vmatrix} 1 & e^{-2x} & e^{3x} \\ 0 & -2e^{-2x} & 3e^{3x} \\ 0 & 0 & 15e^{3x} \end{vmatrix}$$

$$= e^{3x}\begin{vmatrix} -2e^{-2x} & 3e^{3x} \\ 0 & 15e^{3x} \end{vmatrix} = e^{3x}(-30e^{x}) = -30e^{4x}$$

$\neq 0 \ \forall x$

(d) $x, x\ln x, x^2\ln x$; $(0,\infty)$

$R_2 - \frac{1}{x}R_1$

$$W = \begin{vmatrix} x & x\ln x & x^2\ln x \\ 1 & \ln x+1 & 2x\ln x + x \\ 0 & \frac{1}{x} & 2\ln x+3 \end{vmatrix} = \begin{vmatrix} x & x\ln x & x^2\ln x \\ 0 & 1 & x\ln x+x \\ 0 & \frac{1}{x} & 2\ln x+3 \end{vmatrix}$$

$$= x\begin{vmatrix} 1 & x\ln x+x \\ 1/x & 2\ln x+3 \end{vmatrix} = x\left(2\ln x+3-\ln x-1\right)$$

$$= x\left(\ln x+2\right) \neq 0 \text{ on } (0,\infty)$$

((e)) $e^x\cos(2x),\ e^x\sin(2x),\ e^x \ ;\ (-\infty,\infty)$

$$W=\begin{vmatrix} e^x\cos(2x) & e^x\sin(2x) & e^x \\ e^x\cos(2x)-2e^x\sin(2x) & e^x\sin(2x)+2e^x\cos(2x) & e^x \\ \begin{pmatrix} e^x\cos(2x)-2e^x\sin(2x) \\ -2e^x\sin(2x)-4e^x\cos(2x)\end{pmatrix} & \begin{pmatrix} e^x\sin(2x)+2e^x\cos(2x) \\ +2e^x\cos(2x)-4e^x\sin(2x)\end{pmatrix} & e^x \end{vmatrix}$$

factor out an e^x from each row

$$= e^{3x}\begin{vmatrix} \cos(2x) & \sin(2x) & 1 \\ \cos(2x)-2\sin(2x) & \sin(2x)+2\cos(2x) & 1 \\ -3\cos(2x)-4\sin(2x) & -3\sin(2x)+4\cos(2x) & 1 \end{vmatrix}$$

r_2-r_1
r_3-r_1

$$= e^{3x}\begin{vmatrix} \cos(2x) & \sin(2x) & 1 \\ -2\sin(2x) & 2\cos(2x) & 0 \\ -4\cos(2x)-4\sin(2x) & -4\sin(2x)+4\cos(2x) & 0 \end{vmatrix}$$

$$= 8e^{3x}\begin{vmatrix} \sin(2x) & -\cos(2x) \\ \cos(2x)+\sin(2x) & \sin(2x)-\cos(2x) \end{vmatrix}$$

$$= 8e^{3x}\left(\sin^2(2x)-\cancel{\sin(2x)\cos(2x)}+\cos^2(2x)+\cancel{\sin(2x)\cos(2x)}\right)$$

$$= 8e^{3x} \neq 0 \text{ on } \mathbb{R}$$

4 (a) $y'' - y' - 12y = 0 \; ; \{e^{-3x}, e^{4x}\}; (-\infty, \infty)$

- Is e^{-3x} a solution?

$$\left.\begin{aligned} y &= e^{-3x} \\ y' &= -3e^{-3x} \\ y'' &= 9e^{-3x} \end{aligned}\right\} \; y'' - y' - 12y = 9e^{-3x} + 3e^{-3x} - 12e^{-3x} = 0 \quad \underline{\text{Yes}}$$

- Is e^{4x} a solution?

$$\left.\begin{aligned} y &= e^{4x} \\ y' &= 4e^{4x} \\ y'' &= 16e^{4x} \end{aligned}\right\} \Rightarrow y'' - y' - 12y = 16e^{4x} - 4e^{4x} - 12e^{4x} = 0 \quad \underline{\text{Yes}}$$

- Are they linearly independent?

$$W = \begin{vmatrix} e^{-3x} & e^{4x} \\ -3e^{-3x} & 4e^{4x} \end{vmatrix} = 4e^{x} + 3e^{x} = 7e^{x} \neq 0 \quad \underline{\text{Yes}}$$

$\Rightarrow$ Fundamental set

$\Rightarrow$ General solution: $\boxed{y = c_1 e^{-3x} + c_2 e^{4x}}$

(b) $y^{(4)} + y'' = 0 \; ; \{1, x, \cos x, \sin x\}; (-\infty, \infty)$

- Is $y = 1$ a solution? Yes $(y'' = y^{(4)} = 0)$
- Is $y = x$ a solution? Yes $(y'' = y^{(4)} = 0)$
- Is $y = \cos x$ a solution?

$$\left.\begin{aligned} y' &= -\sin x \\ y'' &= -\cos x \\ y''' &= \sin x \\ y^{(4)} &= \cos x \end{aligned}\right\rangle \; y'' + y^{(4)} = 0 \quad \underline{\text{Yes}}$$

- Is $y = \sin x$ a solution?

$$\left.\begin{aligned} y' &= \cos x \\ y'' &= -\sin x \\ y''' &= -\cos x \\ y^{(4)} &= \sin x \end{aligned}\right\rangle \; y'' + y^{(4)} = 0 \quad \underline{\text{Yes}}$$

• Are $\{1, x, \cos x, \sin x\}$ linearly independent?

$$W = \begin{vmatrix} 1 & x & \cos x & \sin x \\ 0 & 1 & -\sin x & \cos x \\ 0 & 0 & -\cos x & -\sin x \\ 0 & 0 & \sin x & -\cos x \end{vmatrix} = \begin{vmatrix} 1 & -\sin x & \cos x \\ 0 & -\cos x & -\sin x \\ 0 & \sin x & -\cos x \end{vmatrix} = \begin{vmatrix} -\cos x & -\sin x \\ \sin x & -\cos x \end{vmatrix} = \cos^2 x + \sin^2 x$$

$= 1$

$\neq 0 \quad \forall x$

$\Rightarrow$ lin. indp.

$\Rightarrow$ fundamental set

$\Rightarrow$ general solution: $\boxed{y = C_1 + C_2 x + C_3 \cos x + C_4 \sin x}$

5 (a) $y'' - y' - 12y = \sin x - 13\cos x$

$$\left.\begin{aligned} y_p &= \cos x \\ y_p' &= -\sin x \\ y_p'' &= -\cos x \end{aligned}\right\} \Rightarrow y_p'' - y_p' - 12y_p = -\cos x + \sin x - 12\cos x = \sin x - 13\cos x$$

✓

$\Rightarrow$ General Solution: $\boxed{y = C_1 e^{-3x} + C_2 e^{4x} + \cos x}$

(b) $y^{(4)} + y'' = 20e^{2x}$

$y_p = e^{2x}$

$y_p' = 2e^{2x}$

$y_p'' = 4e^{2x}$

$y_p''' = 8e^{2x}$

$y_p^{(4)} = 16e^{2x}$

$y_p^{(4)} + y'' = 20e^{2x}$ ✓

$\Rightarrow$ General solution: $\boxed{y = C_1 + C_2 x + C_3 \cos x + C_4 \sin x + e^{2x}}$