

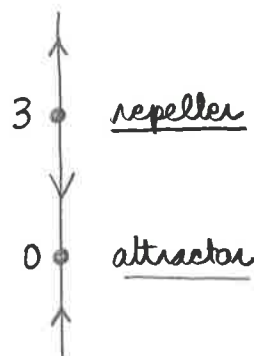
① Attractors & Repellers

(a) $y' = y^2 - 3y$

$f(y) = y^2 - 3y = y(y-3)$

$f(y) = 0 \Rightarrow \boxed{y=0; y=3}$ equilibrium solutions

y		0	3	
$y(y-3)$	+	+	0	-
	+	+	0	+

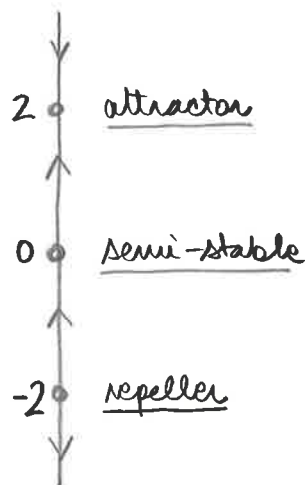


(b) $y' = y^2(4-y^2)$

$f(y) = y^2(4-y^2)$

$f(y) = 0 \Rightarrow \boxed{y=0; y=2; y=-2}$ equilibrium solutions

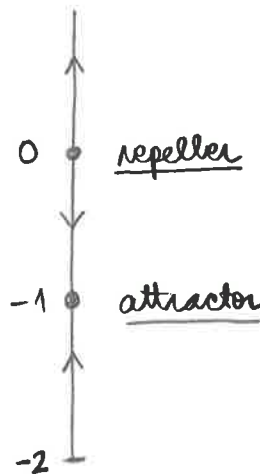
y		-2	0	2	
$4-y^2$	-	0	+	+	0
y^2	+	+	0	+	+
$f(y)$	-	0	+	0	-



(c) $y' = y \ln(y+2)$

$f(y) = y \ln(y+2) = 0 \Rightarrow \boxed{y=0, y=-1}$ equilibrium solns.

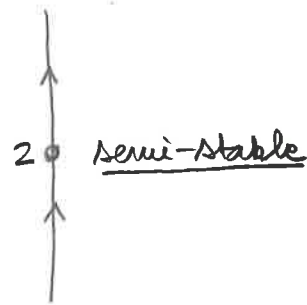
y		-2	-1	0	∞
$\ln(y+2)$	-	0	+	+	+
$f(y)$	+	0	-	0	+



(d) $y' = (y-2)^4$

$f(y) = (y-2)^4 = 0 \Rightarrow \boxed{y=2}$
 equil. sol.

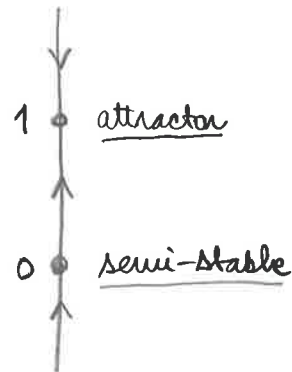
$(y-2)^4 \geq 0$ for all y



(e) $y' = y^2 - y^3$

$f(y) = y^2(1-y) \Rightarrow \boxed{y=0; y=1}$
 equil. sol.

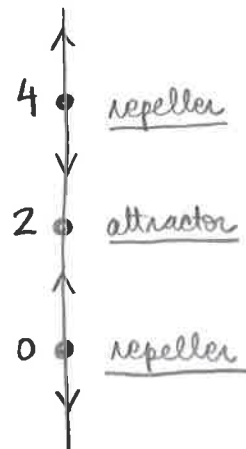
	0	1
$1-y$	+	-
y^2	+	+
$f(y)$	+	-



(f) $y' = y(2-y)(4-y)$

$f(y) = y(2-y)(4-y) \Rightarrow \boxed{y=0; y=2; y=4}$
 equil. sol.

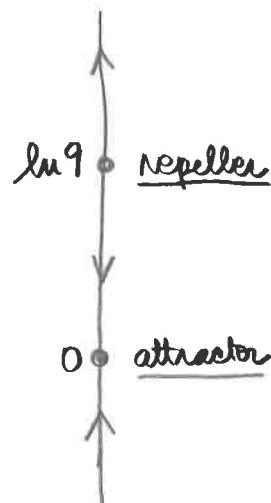
	0	2	4
y	-	+	+
$2-y$	+	+	-
$4-y$	+	+	+
$f(y)$	-	+	-



(g) $y' = \frac{ye^y - 9y}{e^y}$

$f(y) = \frac{y(e^y - 9)}{e^y} = 0 \Rightarrow \boxed{y=0; y=\ln 9}$
 equil. sol.

	0	$\ln 9$
y	-	+
$e^y - 9$	-	+
$f(y)$	+	-



② Logistic Equation:

$$\frac{dy}{dt} = r \left(1 - \frac{1}{K} y\right) y \quad \begin{matrix} r > 0 \\ K > 0 \end{matrix}$$

Solution subject to $y(0) = y_0$:

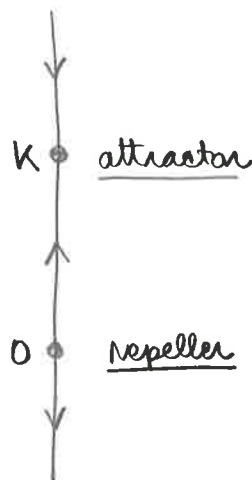
$$y(t) = \frac{Ky_0}{y_0 + (K - y_0)e^{-rt}}; t > 0$$

(a) Equilibrium Solutions:

$$y' = r \left(1 - \frac{1}{K} y\right) y$$

$$y' = 0 \Rightarrow \boxed{y = 0; y = K}$$

	0	K
y	- - 0 + + + +	
$1 - \frac{1}{K} y$	+ + + 0 - -	
y'	- 0 + 0 -	



(b) Suppose $0 < y_0 < K$. Show that $y(t)$ is increasing on $(0, \infty)$, and $0 < y(t) \leq K, \forall t > 0$.

Compute $y'(t)$:

$$\begin{aligned} y'(t) &= - \frac{Ky_0}{(y_0 + (K - y_0)e^{-rt})^2} ((K - y_0) \cdot (-r) e^{-rt}) \\ &= \frac{r y_0 K (K - y_0) e^{-rt}}{(y_0 + (K - y_0)e^{-rt})^2} > 0 \Rightarrow y(t) \text{ is increasing} \end{aligned}$$

It is obvious that $y(t) > 0$ for all t . To prove the other inequality:

$$\lim_{t \rightarrow \infty} y(t) = \lim_{t \rightarrow \infty} \frac{Ky_0}{y_0 + (K - y_0) \underbrace{e^{-rt}}_{\rightarrow 0}} = \frac{Ky_0}{y_0} = K$$

So y is increasing and $\lim_{t \rightarrow \infty} y(t) = K$, so $y(t) \leq K$ for all $t > 0$.

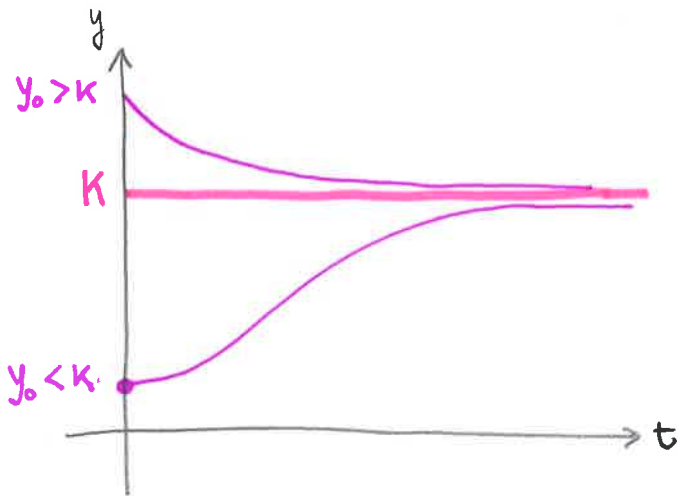
⇒ If the initial population is smaller than the carrying capacity, the population increases over time and approaches the carrying capacity over time.

(c.) Suppose $y_0 > K$. Then

$$y'(t) = \frac{ry_0 K (K - y_0) e^{-rt}}{(y_0 + (K - y_0) e^{-rt})^2} < 0 \Rightarrow y(t) \text{ is } \underline{\text{decreasing}}$$

The limit as $t \rightarrow \infty$ remains $\lim_{t \rightarrow \infty} y(t) = K$, so because y is decreasing $y(t) \geq K$ for all t .

⇒ If the initial population exceeds the carrying capacity, the population decreases, stabilizing at the carrying capacity.



0 only if $y = 0$ or $y = K$, neither of which are possible on this interval.

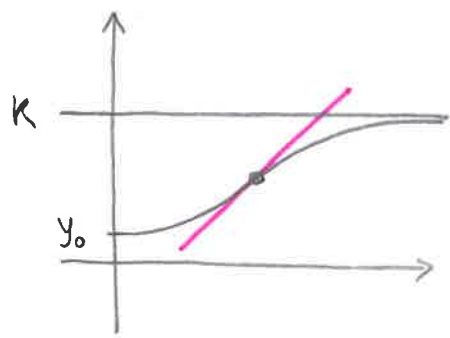
(d) $y' = r(1 - \frac{1}{K}y)y$

$y'' = r(-\frac{1}{K}y + (1 - \frac{1}{K}y)) = r(1 - \frac{2}{K}y)y'$

(Recall that an inflection point of a function $f(x)$ is a point where the second derivative $f''(x)$ changes sign, and f changes concavity).

$y'' = 0 \Rightarrow y = \frac{K}{2} \Rightarrow$ The inflection point occurs when the population reaches $\frac{K}{2}$.

\Rightarrow This can only happen if $y_0 < K$ (because if $y_0 > K$, then $y(t) \geq K$ for all t).



But y'' is the derivative of y' (the rate of growth), so the sign of y'' tells us the monotonicity of y' .

	$\frac{K}{2}$				
y''	+	+	0	-	
y'	↗		max	↘	

\Rightarrow The inflection point is where the rate of growth is maximal.

\Rightarrow The population initially increases steadily from y_0 . Once the population reaches half the carrying capacity, it begins to grow slower and slower, eventually tapering off around K .

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$$y_0 = 3 \text{ mg}$$

$$K = 100 \text{ mg}$$

$$r = 0.2 / \text{hour}$$

$$\frac{dy}{dt} = 0.2y \left(1 - \frac{1}{100}y\right)$$

$$y(t) = \frac{300}{3 + 97e^{-0.2t}}$$

(a) $y(t) = 20 \text{ mg}$

$$\Rightarrow \frac{300}{3 + 97e^{-0.2t}} = 20 \Rightarrow 3 + 97e^{-0.2t} = 15 \Rightarrow e^{-0.2t} = \frac{12}{97}$$

$$\Rightarrow -0.2t = \ln(12/97)$$

$$\Rightarrow t = \frac{\ln(97/12)}{0.2} \approx 10.45 \text{ hours}$$

(b) $y(t) = 200 \text{ mg}$

The long way : $\frac{300}{3 + 97e^{-0.2t}} = 200 \Rightarrow 3 + 97e^{-0.2t} = \frac{3}{2}$

$$\Rightarrow 97e^{-0.2t} = -\frac{3}{2}$$

No solutions : $97e^{-0.2t} > 0 \forall t$

The short way : Never, because 200 mg exceeds the carrying capacity.