Homework 4
– Solutions –
Bernoulli Equations

Bernoulli DE: $\boxed{\frac{dy}{dx} + P(x)y = Q(x)y^{\alpha}}$ $(\alpha \in \mathbb{R})$

$$y^{-\alpha}\frac{dy}{dx} + P(x)y^{1-\alpha} = Q(x)$$

Substitution: $\boxed{u = y^{1-\alpha}}$ $(\alpha \neq 1)$

$$\frac{du}{dx} = (1-\alpha)y^{-\alpha}\frac{dy}{dx} \Rightarrow y^{-\alpha}\frac{dy}{dx} = \frac{1}{1-\alpha}\frac{du}{dx}$$

$$\frac{1}{1-\alpha}\frac{du}{dx} + P(x)u = Q(x)$$

$$\boxed{\frac{du}{dx} + (1-\alpha)P(x)u = (1-\alpha)Q(x)} = \text{linear}$$

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① $x\frac{dy}{dx} + y = y^2x^2\ln x$

$$\frac{dy}{dx} + \frac{1}{x}y = y^2 x \ln x \quad (\text{Bernoulli with } \alpha = 2)$$

$$\boxed{u = \frac{1}{y}} \Rightarrow \frac{du}{dx} - \frac{1}{x}u = -x\ln x$$

Integrating Factor: $\mu(x) = e^{-\int \frac{1}{x}dx} = e^{-\ln|x|} = \frac{1}{|x|}$

Take $\mu(x) = \frac{1}{x}$ $\Rightarrow$ $\frac{d}{dx}\left(\frac{1}{x}u\right) = -\ln x \Rightarrow \frac{1}{x}u = -x\ln x + x + C$

(You can assume $x>0$ here safely – there is a $\ln x$)

$$\Rightarrow u = -x^2\ln x + x^2 + Cx$$

$$\Rightarrow \frac{1}{y} = -x^2\ln x + x^2 + Cx$$

$$\Rightarrow \boxed{y = \frac{1}{-x^2\ln x + x^2 + Cx}}$$

② $\frac{dy}{dx} - \frac{1}{x}y = 4x^2\cos x\,\frac{1}{y}$; $x>0$

Bernoulli with $\alpha = -1 \Rightarrow 1-\alpha = 2 \Rightarrow \boxed{u = y^2}$

$$\left.\begin{aligned} &\frac{du}{dx} - \frac{2}{x}u = 8x^2\cos x \\ &\mu(x) = e^{-2\int\frac{1}{x}dx} = \frac{1}{x^2} \end{aligned}\right\} \Rightarrow \frac{d}{dx}\left(\frac{1}{x^2}u\right) = 8\cos x \Rightarrow \frac{1}{x^2}u = 8\sin x + C$$

$$\Rightarrow u = 8x^2\sin x + Cx^2$$

$$\Rightarrow \boxed{y^2 = 8x^2\sin x + Cx^2}$$

(3) $\frac{dy}{dx} - \frac{3}{2x}y = 6y^{1/3}x^2\ln x$ Bernoulli with $\alpha = 1/3 \Rightarrow 1-\alpha = 2/3$

$\boxed{u = y^{2/3}} \Rightarrow \frac{du}{dx} - \frac{1}{x}u = 4x^2\ln x$

$\mu(x) = \frac{1}{x} \Rightarrow \frac{d}{dx}\left(\frac{1}{x}u\right) = 4x\ln x$

$\Rightarrow \frac{1}{x}u = \int 4x\ln x\,dx = \int 2(x^2)'\ln x\,dx$

$= 2x^2\ln x - \int 2x^2\cdot\frac{1}{x}dx$

$= 2x^2\ln x - x^2 + C$

$\Rightarrow u = x(2x^2\ln x - x^2 + C)$

$\Rightarrow \boxed{y^{2/3} = 2x^3\ln x - x^3 + Cx}$

(4) $y' + 2x^{-1}y = 6y^2x^4$ Bernoulli with $\alpha = 2 \Rightarrow 1-\alpha = -1$

$\boxed{u = y^{-1}} \Rightarrow \frac{du}{dx} - \frac{2}{x}u = -6x^4$

$\mu(x) = \frac{1}{x^2} \Rightarrow \frac{d}{dx}\left(u\frac{1}{x^2}\right) = -6x^2 \Rightarrow u\frac{1}{x^2} = -2x^3 + C$

$\Rightarrow u = x^2(-2x^3 + C)$

$\Rightarrow \boxed{y = \frac{1}{x^2(C - 2x^3)}}$

(5) $y' + 4xy = 4x^3y^{1/2}$ Bernoulli with $\alpha = 1/2 \Rightarrow 1-\alpha = 1/2$

$\boxed{u = \sqrt{y}} \Rightarrow \frac{du}{dx} + 2xu = 2x^3$

$\mu(x) = e^{\int 2x\,dx} = e^{x^2}$

$\Rightarrow \frac{d}{dx}(ue^{x^2}) = 2x^3e^{x^2}$

$\Rightarrow ue^{x^2} = \int 2x^3e^{x^2}dx = \int x^2\cdot 2xe^{x^2}dx = \int x^2(e^{x^2})'dx = x^2e^{x^2} - \int 2xe^{x^2}dx$

$= x^2e^{x^2} - e^{x^2} + C$

$\Rightarrow u = x^2 - 1 + Ce^{-x^2}$

$\Rightarrow \boxed{y = (x^2 - 1 + Ce^{-x^2})^2}$

(6) $(x-1)(x-2)(y'-\sqrt{y})=2y$; $x>2$

$$y'-\sqrt{y}=\frac{2y}{(x-1)(x-2)}$$

$$\frac{dy}{dx}-\frac{2}{(x-1)(x-2)}y=\sqrt{y}$$ Bernoulli with $\alpha=\frac{1}{2} \Rightarrow 1-\alpha=\frac{1}{2}$

$$\boxed{u=\sqrt{y}} \Rightarrow \frac{du}{dx}-\frac{1}{(x-1)(x-2)}u=\frac{1}{2}$$

$$\int p(x)\,dx=-\int\frac{1}{(x-1)(x-2)}dx=\int\frac{(x-2)-(x-1)}{(x-1)(x-2)}dx=\ln|x-1|-\ln|x-2|$$
$$=\ln\frac{x-1}{x-2}$$

$$\Rightarrow \mu(x)=\frac{x-1}{x-2}$$

$$\Rightarrow \frac{d}{dx}\left(u\,\frac{x-1}{x-2}\right)=\frac{1}{2}\,\frac{x-1}{x-2} \Rightarrow u\,\frac{x-1}{x-2}=\frac{1}{2}\int\frac{x-1}{x-2}dx$$
$$=\frac{1}{2}\int\frac{x-2+1}{x-2}dx=\frac{1}{2}\int\left(1+\frac{1}{x-2}\right)dx$$
$$=\frac{1}{2}\left(x+\ln(x-2)+c\right)$$

$$\Rightarrow u=\frac{1}{2}\,\frac{x-2}{x-1}\left(x+\ln(x-2)+c\right)$$

$$\Rightarrow \boxed{y=\frac{1}{4}\left(\frac{x-2}{x-1}\right)^2\left(x+\ln(x-2)+c\right)^2}$$

(7) $\frac{dy}{dx}-\frac{1}{(\pi-1)x}y=\frac{3}{1-\pi}xy^{\pi}$ Bernoulli with $\alpha=\pi \Rightarrow 1-\alpha=1-\pi$

$$\boxed{u=y^{1-\pi}} \Rightarrow \frac{du}{dx}+\frac{1}{x}u=3x$$

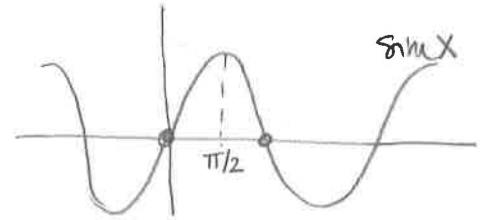
$$\mu(x)=x \Rightarrow \frac{d}{du}(ux)=3x^2 \Rightarrow ux=x^3+c \Rightarrow u=x^2+\frac{c}{x}$$

$$\Rightarrow \boxed{y=\left(x^2+\frac{c}{x}\right)^{\frac{1}{1-\pi}}}$$

8 $y' + y\cot x = y^3 \sin^3 x \; ; \; y(\pi/2) = 1$ (IVP) Bernoulli with $\alpha = 3$
$\Rightarrow 1-\alpha = -2$

$\boxed{u = y^{-2}} \Rightarrow \dfrac{du}{dx} - 2(\cot x)u = -2\sin^3 x$

$$\int p(x)\,dx = -2\int \cot x\,dx = -2\int \frac{\cos x}{\sin x}\,dx$$
$$= -2\ln|\sin x|$$



The ODE contains the function cot x, which is undefined whenever $\sin x = 0$
The initial condition is given at $x = \pi/2$, so you can assume you are working on $x \in (0, \pi)$
(where $\sin x > 0$)

$\Rightarrow \int p(x)\,dx = -2\ln(\sin x)$

$\Rightarrow \mu(x) = \dfrac{1}{\sin^2 x}$

$\Rightarrow \dfrac{d}{dx}\left(u\,\dfrac{1}{\sin^2 x}\right) = -2\sin x \Rightarrow u\,\dfrac{1}{\sin^2 x} = 2\cos x + C$

$\Rightarrow u = \sin^2 x\,(2\cos x + C)$

$\Rightarrow \dfrac{1}{y^2} = \sin^2 x\,(2\cos x + C)$

IVP : $x = \pi/2,\ y = 1 \Rightarrow 1 = C \Rightarrow \boxed{\dfrac{1}{y^2} = \sin^2 x\,(2\cos x + 1)}$

9 $2\dfrac{dy}{dx} + (\tan x)y = \dfrac{(4x+5)^2}{\cos x}y^3$

$\dfrac{dy}{dx} + \dfrac{1}{2}(\tan x)y = \dfrac{(4x+5)^2}{2\cos x}y^3$ Bernoulli with $\alpha = 3 \Rightarrow 1-\alpha = -2$

$\boxed{u = y^{-2}} \Rightarrow \dfrac{du}{dx} - (\tan x)u = -\dfrac{(4x+5)^2}{\cos x}$

$$\int p(x)\,dx = -\int \tan x\,dx = \int \frac{-\sin x}{\cos x}\,dx$$
$$= \ln|\cos x|$$

$\Rightarrow \boxed{\mu(x) = \cos x}$

$\Rightarrow \dfrac{d}{dx}(u\cos x) = -(4x+5)^2$

$\Rightarrow u\cos x = -\dfrac{1}{12}(4x+5)^3 + C$

$\Rightarrow \boxed{\dfrac{1}{y^2} = \dfrac{1}{\cos x}\left(-\dfrac{1}{12}(4x+5)^3 + C\right)}$