1) $f(x,y) = x^3 + 2xy^2 - \dfrac{y^4}{x}$

Quick version : homogeneous of degree 3 by inspection of degrees

Long version : $f(\alpha x, \alpha y) = (\alpha x)^3 + 2(\alpha x)(\alpha y)^2 - \dfrac{(\alpha y)^4}{\alpha x}$

$$= \alpha^3\left(x^3 + 2xy^2 - \frac{y^4}{x}\right) = \alpha^3 f(x,y).$$

2) $f(x,y) = \left(\dfrac{2}{x} + \dfrac{3}{y}\right)^2$

Quick version : homogeneous of degree -2 by inspection of degrees

Long version : $f(\alpha x, \alpha y) = \left(\dfrac{2}{\alpha x} + \dfrac{3}{\alpha y}\right)^2 = \dfrac{1}{\alpha^2}\left(\dfrac{2}{x} + \dfrac{3}{y}\right)^2 = \alpha^{-2} f(x,y)$

3) $\ln(x^2) - 2\ln y$

Inspection of degrees does not work so well on functions that are not simply sums of polynomials and/or rational functions. At a glance, this looks like it is probably not homogeneous, but let's check :

$$f(\alpha x, \alpha y) = \ln(\alpha^2 x^2) - 2\ln(\alpha y) = \cancel{\ln(\alpha^2)} + \ln(x^2) \underbrace{- 2\ln\alpha}_{-\cancel{\ln(\alpha^2)}} - 2\ln y = f(x,y)$$

=> homogeneous of degree 0

4) $\ln(x^2) - \ln y$

$$f(\alpha x, \alpha y) = \ln(\alpha^2 x^2) - \ln(\alpha y) = \ln(\alpha^2) + \ln(x^2) - \ln\alpha - \ln y$$

$$= \ln(\alpha^2) - \ln\alpha + f(x,y)$$ => not homogeneous

5) $\cos\left(\dfrac{x^2}{x+y}\right)$

$$f(\alpha x, \alpha y) = \cos \frac{\alpha^2 x^2}{\alpha(x+y)}$$

$$= \cos \frac{\alpha x^2}{(x+y)}$$

=> not homogeneous

6) $\sin\left(\dfrac{x}{x+y}\right)$

$$f(\alpha x, \alpha y) = \sin\left(\frac{\alpha x}{\alpha(x+y)}\right)$$

$$= \sin\left(\frac{x}{x+y}\right)$$

=> homogeneous of degree 0

⑦ $(x+y)dx+xdy=0$

- separable? No
- exact? Yes: $M_y=N_x=1$
- homogeneous? Yes: M, N homogeneous of degree 1.

Solution 1: (Exactness)

Potential: $\frac{\partial f}{\partial x}=x+y \Rightarrow f(x,y)=\frac{x^2}{2}+xy+g(y)$

$\Rightarrow \frac{\partial f}{\partial y}=x+g'(y) = x$

$\Rightarrow f(x,y)=\frac{x^2}{2}+xy$

$\boxed{x^2+2xy=c}$

Solution 2: (Homogeneity) I

$y=ux \Rightarrow (x+ux)dx+x(udx+xdu)=0$

$dy=udx+xdu$

$(1+2u)dx+xdu=0$

$\frac{1}{x}dx+\frac{1}{1+2u}du=0$

$\ln|x|+\frac{1}{2}\ln|1+2u|=c$

$\ln\left(|x|\sqrt{|1+2u|}\right)=c \Rightarrow x\sqrt{|1+2u|}=c$

$\Rightarrow x^2(1+2u)=c$

$\Rightarrow x^2\left(1+2\frac{y}{x}\right)=c \Rightarrow \boxed{x^2+2xy=c}$

Solution 3: (Homogeneity) II

$(x+y)dx+xdy=0$

$x+y+x\frac{dy}{dx}=0$

$1+\frac{y}{x}+\frac{dy}{dx}=0$

$u=\frac{y}{x}$ $\Rightarrow 1+u+\frac{xdu+udx}{dx}=0$

$\Rightarrow 1+u+x\frac{du}{dx}+u=0$

$\Rightarrow 1+2u+x\frac{du}{dx}=0$

$\Rightarrow$ Same: $\frac{1}{x}dx+\frac{1}{1+2u}du=0.$

⑧ $(y^2+yx)\,dx - x^2\,dy = 0$

- separable? No.
- exact? No. $M_y = 2y+x$; $N_x = -2x$
- homogeneous? Yes - degree 2

$y = ux \Rightarrow (u^2x^2 + ux^2)\,dx - x^2(u\,dx + x\,du) = 0$

$(u^2+u)\,dx - u\,dx - x\,du = 0$

$u^2\,dx = x\,du$

$\frac{1}{x}\,dx = \frac{1}{u^2}\,du \Rightarrow \ln|x| = -\frac{1}{u} + C$

$\Rightarrow \ln|x| = -\frac{x}{y} + C \Rightarrow x = Ce^{-x/y}$

⑨ $\frac{dy}{dx} = \frac{y-x}{y+x}$

$\left.\begin{array}{l} \frac{dy}{dx} = \frac{y/x - 1}{y/x + 1} \\ u = y/x \end{array}\right\} \Rightarrow \frac{x\,du + u\,dx}{dx} = \frac{u-1}{u+1} \Rightarrow x\frac{du}{dx} + u = \frac{u-1}{u+1}$

$\Rightarrow x\frac{du}{dx} = -\frac{u^2+1}{u+1}$

$\Rightarrow \frac{1}{x}\,dx = -\frac{u+1}{u^2+1}\,du$

$\int \frac{u+1}{u^2+1}\,du = \int \frac{u}{u^2+1}\,du + \int \frac{1}{u^2+1}\,du$

$= \frac{1}{2}\ln(u^2+1) + \arctan u$

$\Rightarrow \ln|x| + \frac{1}{2}\ln(u^2+1) + \arctan(u) = C$

$\Rightarrow \ln\left(|x|\sqrt{\frac{y^2}{x^2}+1}\right) + \arctan\left(\frac{y}{x}\right) = C$

$\Rightarrow \boxed{\ln\left(\sqrt{x^2+y^2}\right) + \arctan\left(\frac{y}{x}\right) = C}$

or, simpler $\boxed{\ln(x^2+y^2) + 2\arctan(y/x) = C}$

⑩ $\frac{dy}{dx} = \frac{y}{x} + \frac{x}{y}$

$u = y/x \Rightarrow \frac{x\,du + u\,dx}{dx} = u + \frac{1}{u}$

$\Rightarrow x\frac{du}{dx} + u = u + \frac{1}{u} \Rightarrow x\frac{du}{dx} = \frac{1}{u} \Rightarrow u\,du = \frac{1}{x}\,dx$

$\Rightarrow \ln|x| = \frac{1}{2}u^2 + C$

$= \frac{1}{2}\frac{y^2}{x^2} + C$

$\Rightarrow \boxed{x = Ce^{\frac{1}{2}y^2/x^2}}$

$$\text{(11)}\quad 2x^2y\,dx = (3x^3+y^3)dy$$

– separable? No
– exact? $M_y = 2x^2$; $N_x = -9x^2$ No.
– homogeneous? Yes – degree 3.

$y = ux \Rightarrow$

$$2ux^3dx = (3x^3+u^3x^3)(u\,dx + x\,du)$$

$$2u\,dx = (3+u^3)(u\,dx+x\,du)$$

$$(2u-3u-u^4)dx = (3+u^3)x\,du$$

$$\frac{1}{x}dx = -\frac{3+u^3}{u+u^4}du$$

$$\frac{3+u^3}{u+u^4} = \frac{3+u^3}{u(1+u^3)} = \frac{1}{u} + \frac{2}{u(1+u^3)} = \frac{1}{u} + 2\left(\frac{1}{u} - \frac{u^2}{1+u^3}\right) = \frac{3}{u} - \frac{2u^2}{1+u^3}$$

$$\frac{1}{u} - \frac{u^2}{1+u^3} = \frac{1+\cancel{u^3}-\cancel{u^3}}{u(1+u^3)}$$

$$\Rightarrow \int \frac{3+u^3}{u+u^4}du = 3\ln|u| - \frac{2}{3}\ln|1+u^3|$$

$$\ln|x| + 3\ln|u| - \frac{2}{3}\ln|1+u^3| = c$$

$$\ln\left|x u^3 \frac{1}{(1+u^3)^{2/3}}\right| = c$$

$$x\frac{y^3}{x^3}\frac{1}{\left(1+\frac{y^3}{x^3}\right)^{2/3}} = c$$

$$\frac{y^3}{\cancel{x^2}}\frac{\cancel{x^2}}{(x^3+y^3)^{2/3}} = c$$

$y^3 = c(x^3+y^3)^{2/3}$ (raise both sides to power 3)

$$\boxed{y^9 = c(x^3+y^3)^2}$$

(12) $\frac{dy}{dx} = \frac{y}{x} + \frac{x^2}{y^2} + 1$

$u = \frac{y}{x} \Rightarrow \frac{x\,du + u\,dx}{dx} = u + \frac{1}{u^2} + 1$

$\Rightarrow x\frac{du}{dx} + \cancel{u} = \cancel{u} + \frac{1}{u^2} + 1 \Rightarrow \frac{1}{x}dx = \frac{u^2}{1+u^2}du$

$\Rightarrow \ln|x| = \int \frac{(u^2+1)-1}{u^2+1}du = u - \arctan(u) + C$

$\Rightarrow \boxed{\ln|x| = \frac{y}{x} - \arctan\left(\frac{y}{x}\right) + C}$ or $\boxed{x = C\,e^{y/x - \arctan(y/x)}}$

(13) $y\frac{dx}{dy} = x + 4ye^{-2x/y} \Rightarrow \frac{dx}{dy} = \frac{x}{y} + 4e^{-2x/y}$

$u = \frac{x}{y}$

$\frac{u\,dy + y\,du}{dy} = u + 4e^{-2u}$

$\cancel{u} + y\frac{du}{dy} = \cancel{u} + 4e^{-2u}$

$\frac{1}{y}dy = \frac{1}{4}e^{2u}du \Rightarrow \ln|y| = \frac{1}{8}e^{2u} + C$

$\Rightarrow \boxed{\ln|y| = \frac{1}{8}e^{2x/y} + C}$ or $\boxed{e^{2x/y} = \ln(y^8) + C}$

(14) $\frac{dy}{dx} = \frac{y}{x}\ln\frac{y}{x}$

$u = \frac{y}{x} \Rightarrow u + x\frac{du}{dx} = u\ln u$

$\Rightarrow x\frac{du}{dx} = u(\ln u - 1)$

$\frac{1}{x}dx = \frac{1}{u(\ln u - 1)}du$

$\ln|x| = \ln|\ln u - 1| + C$

$x = C(\ln u - 1)$

$\ln u - 1 = Cx$

$\ln u = Cx + 1 \Rightarrow u = e^{Cx+1} \Rightarrow \boxed{y = xe^{Cx+1}}$

15 $y\,dx + x(\ln x - \ln y - 1)\,dy = 0 \;;\; y(1) = e$

$$y\,dx + x\left(\ln\frac{x}{y} - 1\right)dy = 0$$

$$\frac{y}{x} + \left(\ln\frac{x}{y} - 1\right)\frac{dy}{dx} = 0$$

$$\frac{y}{x} - \left(1 + \ln\frac{y}{x}\right)\frac{dy}{dx} = 0 \Rightarrow \boxed{\frac{dy}{dx} = \frac{y/x}{1 + \ln y/x}}$$

$$u = \frac{y}{x} \Rightarrow u + x\frac{du}{dx} = \frac{u}{1+\ln u} \Rightarrow x\frac{du}{dx} = \frac{\cancel{u} - \cancel{u} - u\ln u}{1 + \ln u}$$

$$\Rightarrow -\frac{1+\ln u}{u\ln u}\,du = \frac{1}{x}\,dx$$

$$\Rightarrow \int \frac{1}{x}\,dx + \int \frac{1+\ln u}{u\ln u}\,du = 0$$

because $x>0$ (from original eqn.)

$$\Rightarrow \ln|x| + \int \frac{1}{u}\frac{1}{\ln u}\,du + \int \frac{1}{u}\,du = c$$

$$\Rightarrow \ln x + \ln|\ln u| + \ln u = c$$

$$\Rightarrow \ln(ux|\ln u|) = c$$

$$\Rightarrow ux|\ln u| = c \Rightarrow \ln u = \frac{c}{ux} \Rightarrow \ln\left(\frac{y}{x}\right) = \frac{c}{y} \Rightarrow \frac{y}{x} = e^{c/y}$$

$$\Rightarrow \boxed{x = ye^{c/y}}$$

$$x = 1;\; y = e \Rightarrow 1 = e \cdot e^{c/e} \Rightarrow 1 + \frac{c}{e} = 0 \Rightarrow c = -e$$

$$\boxed{x = ye^{-e/y}}$$

(16) $y^2dx + (x^2+xy+y^2)dy = 0 \; ; \; y(0)=1$

$$y^2 + (x^2+xy+y^2)\frac{dy}{dx} = 0$$

$$\left(\frac{y}{x}\right)^2 + \left(1+\frac{y}{x}+\left(\frac{y}{x}\right)^2\right)\frac{dy}{dx} = 0$$

$u = \frac{y}{x}$ $\Rightarrow$ $u^2 + (1+u+u^2)\left(u + x\frac{du}{dx}\right) = 0$

$$(u^2+u+u^2+u^3) + x(1+u+u^2)\frac{du}{dx} = 0$$

$$(u+2u^2+u^3)dx + x(1+u+u^2)du = 0$$

$$\frac{1}{x}dx + \frac{1+u+u^2}{u+2u^2+u^3}du = 0$$

$$= \frac{1+u+u^2}{u(1+u)^2} = \frac{(1+u)^2-u}{u(1+u)^2} = \frac{1}{u} - \frac{1}{(1+u)^2}$$

$\Rightarrow \ln|x| + \ln|u| + \frac{1}{1+u} = c$

$\Rightarrow \ln|y| = c - \frac{1}{1+y/x}$ $\Rightarrow$ $\boxed{y = ce^{-\frac{x}{x+y}}}$

$x=0,\, y=1 \Rightarrow 1 = c$ $\quad \boxed{y = e^{-\frac{x}{x+y}}}$

(17) $(y^2-2xy)dx + x^2dy = 0$

$$\left(\frac{y}{x}\right)^2 - 2\frac{y}{x} + \frac{dy}{dx} = 0$$

$u = \frac{y}{x}$ $\Rightarrow$ $u^2 - 2u + u + x\frac{du}{dx} = 0$

$$(u^2-u)dx + x\,du = 0$$

$$\frac{1}{x}dx + \frac{1}{u^2-u}du = 0$$

$$\ln|x| + \ln|u-1| - \ln|u| = C$$

$$\ln\left|x\left(1-\frac{x}{y}\right)\right| = c$$

$$x - \frac{x^2}{y} = c$$

$$\frac{x^2}{y} = x + c \qquad \boxed{y = \frac{x^2}{x+c}}$$

$$\frac{1}{u^2-u} = \frac{1}{u(u-1)} = \frac{u-(u-1)}{u(u-1)} = \frac{1}{u-1} - \frac{1}{u}$$

(since choosing $u = y/x$ would result in having to integrate e^{1/u^2}, it seems more convenient to use $u = x/y$. Then we would have to integrate terms with e^{u^2} ~> seems easier)

(18) $\left(y^2 + y^2 e^{(x/y)^2} + 2x^2 e^{(x/y)^2}\right) y' = 2xy\, e^{(x/y)^2} \quad /y^2$

$$\left(1 + e^{(x/y)^2} + 2\left(\frac{x}{y}\right)^2 e^{(x/y)^2}\right)\frac{dy}{dx} = 2\left(\frac{x}{y}\right)e^{(x/y)^2}$$

$$\Rightarrow \frac{dx}{dy} = \frac{1 + e^{(x/y)^2} + 2\left(\frac{x}{y}\right)^2 e^{(x/y)^2}}{2\left(\frac{x}{y}\right)e^{(x/y)^2}}$$

$$u = \frac{x}{y} \Rightarrow u + y\frac{du}{dy} = \frac{1 + e^{u^2} + 2u^2 e^{u^2}}{2u e^{u^2}}$$

$$\Rightarrow y\frac{du}{dy} = \frac{1 + e^{u^2}}{2u e^{u^2}} \Rightarrow \frac{1}{y}dy = \frac{2u e^{u^2}}{1 + e^{u^2}}du$$

$$\Rightarrow \ln|y| = \ln\left(1 + e^{u^2}\right) + c$$

$$\Rightarrow \boxed{y = c\left(1 + e^{(x/y)^2}\right)}$$

(19) $(x^2 - y^2)dy = 2xy\, dx$

$$(x^2 - y^2)\frac{dy}{dx} = 2xy$$

$$\left(1 - \frac{y^2}{x^2}\right)\frac{dy}{dx} = 2\frac{y}{x}$$

$$u = \frac{y}{x} \Rightarrow (1 - u^2)\left(u + x\frac{du}{dx}\right) = 2u$$

$$u - u^3 + x(1 - u^2)\frac{du}{dx} = 2u$$

$$x(1 - u^2)\frac{du}{dx} = u^3 + u$$

$$\frac{1 - u^2}{u(1 + u^2)}du = \frac{1}{x}dx$$

$$\ln|x| = \ln|u| - \ln(1 + u^2) + c$$

$$\ln\left|\frac{x}{u}(1 + u^2)\right| = c$$

$$\frac{x^2 + y^2}{y} = c \qquad \boxed{x^2 + y^2 = cy}$$

$$\int \frac{1 - u^2}{u(1 + u^2)}du = \int \frac{1 + u^2 - 2u^2}{u(1 + u^2)}du$$

$$= \int \frac{1}{u} - \frac{2u}{(1 + u^2)}du$$

$$= \ln|u| - \ln(1 + u^2) + c$$

(20) $x\,dy = \left(y + \sqrt{x^2 - y^2}\right) dx$

((a)) $\dfrac{dy}{dx} = \dfrac{y}{x} + \underbrace{\dfrac{1}{x}\sqrt{x^2-y^2}}$

$\dfrac{1}{x}\sqrt{x^2(1-y^2/x^2)} = \dfrac{|x|}{x}\sqrt{1-y^2/x^2}$ → $+\sqrt{1-(y/x)^2}$ if $x>0$; $-\sqrt{1-(y/x)^2}$ if $x<0$

Remember that $\sqrt{x^2} = |x|$, so we need to treat the cases $x>0$ and $x<0$ separately

Case 1: $x>0$ $\quad \dfrac{dy}{dx} = \dfrac{y}{x} + \sqrt{1-(y/x)^2}$

$u = y/x \Rightarrow \cancel{u} + x\dfrac{du}{dx} = \cancel{u} + \sqrt{1-u^2} \Rightarrow \dfrac{1}{\sqrt{1-u^2}}\,du = \dfrac{1}{x}\,dx$

$\Rightarrow \ln|x| = \arcsin(u) + C$

$\Rightarrow \ln x = \arcsin(y/x) + C \Rightarrow \boxed{x = e^{\arcsin(y/x)+C}}$ $x>0$

Case 2: $x<0$ $\quad \dfrac{dy}{dx} = \dfrac{y}{x} - \sqrt{1-(y/x)^2}$

$u = y/x \Rightarrow \cancel{u} + x\dfrac{du}{dx} = \cancel{u} - \sqrt{1-u^2} \Rightarrow -\dfrac{1}{\sqrt{1-u^2}}\,du = \dfrac{1}{x}\,dx$

$\Rightarrow \ln|x| = -\arcsin(u) + C$

$\Rightarrow \ln(-x) = -\arcsin(y/x) + C \Rightarrow \boxed{-x = e^{-\arcsin(y/x)+C}}$ $x<0$

((b)) How can we bring the two formulas together?

$x = e^{\arcsin(y/x)+C}$, $x>0$

$\boxed{-x = e^{-\arcsin(y/x)+C}}$, $x<0$ $\xrightarrow{\text{If } x<0:}$ $|x| = e^{\arcsin(y/-x)+C}$, $x<0$

If $x<0$: $|x| = -x$ & (arcsin) is an odd function

$= e^{\arcsin(y/|x|)+C}$, $x<0$

$$\boxed{|x| = e^{\arcsin(y/|x|)+C}\ ;\ x \neq 0}$$

(c) $y(1)=0$

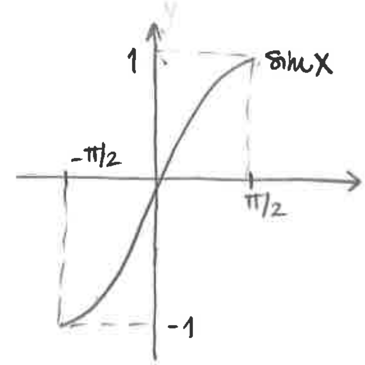
We use the solution expression for $x>0$, since the initial conditions are given at $x=1$, $y=0$.

$$\left.\begin{array}{l} x=e^{\arcsin(y/x)+c} \\ x=1,\ y=0 \end{array}\right\} \Rightarrow 1=e^{\arcsin(0)+c} \Rightarrow 1=e^{c} \Rightarrow \boxed{c=0}$$

$$\boxed{x=e^{\arcsin(y/x)}} \quad x>0$$

(d) First off, recall that

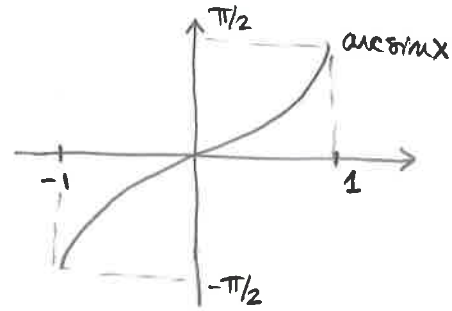
$\arcsin: [-1,1] \to [-\pi/2, \pi/2]$



So $\arcsin(y/x) \in [-\pi/2, \pi/2]$

$$x=e^{\arcsin(y/x)} \Rightarrow \boxed{x\in\left[e^{-\pi/2}, e^{\pi/2}\right]}$$

Now:



$$\ln x = \arcsin(y/x)$$

$$\boxed{\sin(\ln x) = y/x}$$

$$\boxed{y = x\sin(\ln x)}$$

Verify that this satisfies the equation: $xy' = y+\sqrt{x^2-y^2}$ for $x\in\left(e^{-\pi/2}, e^{\pi/2}\right)$

$$y' = \sin(\ln x) + x\cos(\ln x)\cdot\frac{1}{x}$$

$$y' = \sin(\ln x)+\cos(\ln x) \Rightarrow xy' = \underbrace{x\sin(\ln x)}_{y} + x\cos(\ln x)$$

$$\Rightarrow \boxed{xy' = y + x\cos(\ln x)}$$

$$\sqrt{x^2-y^2} = \sqrt{x^2-x^2\sin^2(\ln x)} = \sqrt{x^2\cos^2(\ln x)} = |x|\cdot|\cos(\ln x)| = x\cos(\ln x)$$

because $x>0$

because $x\in\left[e^{-\pi/2}, e^{\pi/2}\right]$

$\Rightarrow \ln x \in [-\pi/2, \pi/2]$

$\Rightarrow \cos(\ln x) \in [0,1]$.

$$\boxed{\underline{\text{Interval}}:\ x\in\left(e^{-\pi/2}, e^{\pi/2}\right)}$$

(e) $y(-1)=0$

Use expression for $x<0$: $\quad -x = e^{-\arcsin(y/x)+c}$

$$x=-1,\ y=0 \Rightarrow 1 = e^{c} \Rightarrow \boxed{c=0}$$

$$\boxed{-x = e^{-\arcsin(y/x)}} \quad x<0$$

$$\Rightarrow \ln(-x) = -\arcsin(y/x)$$

$$\Rightarrow \sin(\ln(-x)) = -y/x$$

$$\Rightarrow \boxed{y = -x\sin(\ln(-x))}$$

$$\Rightarrow \boxed{x \in \left[-e^{\pi/2}, -e^{-\pi/2}\right]}$$

Check solution on this interval:

$$y' = -\sin(\ln(-x)) - x\cos(\ln(-x))\cdot\frac{1}{x}$$

$$\Rightarrow xy' = \underbrace{-x\sin(\ln(-x))}_{y} - x\cos(\ln(-x))$$

$$\Rightarrow xy' = y - x\cos(\ln(-x))$$

$$\sqrt{x^2-y^2} = \sqrt{x^2\cos^2(\ln(-x))} = |x|\,|\cos(\ln(-x))| = -x\cos(\ln(-x))$$

$(-x)$ b/c $x<0$ $\qquad (\cos(\ln(-x)))$

$$x \in \left[-e^{\pi/2}, -e^{-\pi/2}\right]$$

$$\Rightarrow -x \in \left[e^{-\pi/2}, e^{\pi/2}\right]$$

$$\Rightarrow \ln(-x) \in [-\pi/2, \pi/2]$$

$$\Rightarrow \cos(\ln(-x)) \in [0,1]$$

$$\Rightarrow xy' = y + \sqrt{x^2-y^2} \ \checkmark$$

$$\boxed{\underline{\text{Interval}}:\ x \in \left(-e^{\pi/2}, -e^{-\pi/2}\right)}$$

Both solutions in (d). & (e). are part of the same function, $y = |x|\sin(\ln|x|)$ but this function is a solution to the IVPs only on certain intervals.

(See next page for some graphs).