

① $(5x+4y)dx + (4x-8y^3)dy = 0$

$$M(x,y) = 5x+4y \Rightarrow \frac{\partial M}{\partial y} = 4 \quad \checkmark \quad \text{Exact}$$

$$N(x,y) = 4x-8y^3 \Rightarrow \frac{\partial N}{\partial x} = 4 \quad \checkmark$$

Find $f(x,y)$ such that $\frac{\partial f}{\partial x} = 5x+4y$; $\frac{\partial f}{\partial y} = 4x-8y^3$

$$\frac{\partial f}{\partial x} = 5x+4y \Rightarrow f(x,y) = 5\frac{x^2}{2} + 4xy + g(y)$$

$$\Rightarrow \left. \begin{aligned} \frac{\partial f}{\partial y} &= 4x + g'(y) \\ &= 4x - 8y^3 \end{aligned} \right\} \Rightarrow \begin{aligned} g'(y) &= -8y^3 \\ \Rightarrow g(y) &= -2y^4 \end{aligned}$$

$$\Rightarrow f(x,y) = 5\frac{x^2}{2} + 4xy - 2y^4$$

$$\Rightarrow \text{Solutions to the ODE: } \boxed{5\frac{x^2}{2} + 4xy - 2y^4 = C}$$

② $(\sin y - y \sin x)dx + (\cos x + x \cos y - y)dy = 0$

$$M(x,y) = \sin y - y \sin x \Rightarrow \frac{\partial M}{\partial y} = \cos y - \sin x \quad \checkmark \quad \text{Exact}$$

$$N(x,y) = \cos x + x \cos y - y \Rightarrow \frac{\partial N}{\partial x} = -\sin x + \cos y \quad \checkmark$$

Find $f(x,y)$ s.t. $\frac{\partial f}{\partial x} = \sin y - y \sin x$; $\frac{\partial f}{\partial y} = \cos x + x \cos y - y$

$$\frac{\partial f}{\partial x} = \sin y - y \sin x \Rightarrow f(x,y) = x \sin y + y \cos x + g(y)$$

$$\Rightarrow \left. \begin{aligned} \frac{\partial f}{\partial y} &= x \cos y + \cos x + g'(y) \\ &= x \cos y + \cos x - y \end{aligned} \right\} \Rightarrow \begin{aligned} g'(y) &= -y \\ \Rightarrow g(y) &= -\frac{y^2}{2} \end{aligned}$$

$$\Rightarrow f(x,y) = x \sin y + y \cos x - \frac{y^2}{2}$$

$$\Rightarrow \text{Solutions to the ODE: } \boxed{x \sin y + y \cos x - \frac{y^2}{2} = C}$$

$$\textcircled{3} \quad \underbrace{[\cos(xy) - xy \sin(xy)]}_{M(x,y)} dx - \underbrace{x^2 \sin(xy)}_{N(x,y)} dy = 0$$

$$\begin{aligned} \frac{\partial M}{\partial y} &= -x \sin(xy) - x \sin(xy) - x^2 y \cos(xy) \\ &= -2x \sin(xy) - x^2 y \cos(xy) \quad \checkmark \end{aligned}$$

$$\frac{\partial N}{\partial x} = -2x \sin(xy) - x^2 y \cos(xy) \quad \checkmark$$

Exact

Find potential: $\frac{\partial f}{\partial x} = M$; $\frac{\partial f}{\partial y} = N$

$$\frac{\partial f}{\partial x} = \cos(xy) - xy \sin(xy)$$

$$\frac{\partial f}{\partial y} = -x^2 \sin(xy) \quad \rightsquigarrow \text{easier to integrate dy}$$

$$\hookrightarrow \Rightarrow f(x,y) = x \cos(xy) + g(x)$$

$$\Rightarrow \left. \begin{aligned} \frac{\partial f}{\partial x} &= \cos(xy) - xy \sin(xy) + g'(x) \\ &= \cos(xy) - xy \sin(xy) \end{aligned} \right\} \Rightarrow g'(x) = 0 \Rightarrow g(x) = C$$

$$\Rightarrow \boxed{f(x,y) = x \cos(xy)}$$

$$\Rightarrow \text{Solution to ODE: } \boxed{x \cos(xy) = C}$$

$$\textcircled{4} \quad ye^{xy} dx + (2y - xe^{xy}) dy = 0$$

$$M(x,y) = ye^{xy} \Rightarrow \frac{\partial M}{\partial y} = e^{xy} + xy e^{xy}$$

$$N(x,y) = 2y - xe^{xy} \Rightarrow \frac{\partial N}{\partial x} = -e^{xy} - xy e^{xy}$$

} NOT Exact

$$\textcircled{5} \quad (1 + \ln(xy)) dx + xy^{-1} dy = 0$$

$$M(x,y) = 1 + \ln(xy) \Rightarrow \frac{\partial M}{\partial y} = \frac{1}{xy} \cdot x = \frac{1}{y} \quad \checkmark$$

$$N(x,y) = \frac{x}{y} \Rightarrow \frac{\partial N}{\partial x} = \frac{1}{y} \quad \checkmark$$

Exact

Find potential $f(x,y)$:

$$\frac{\partial f}{\partial x} = 1 + \ln(xy)$$

$$\frac{\partial f}{\partial y} = \frac{x}{y} \rightsquigarrow \Rightarrow f(x,y) = x \ln y + g(x)$$

$$\Rightarrow \left. \begin{aligned} \frac{\partial f}{\partial x} &= \ln y + g'(x) \\ &= 1 + \ln(xy) \end{aligned} \right\} \Rightarrow \ln y + g'(x) = 1 + \ln(xy)$$

$$\Rightarrow g'(x) = 1 + \ln(xy) - \ln y = 1 + \ln x$$

$$\Rightarrow g(x) = x + x \ln x - x = x \ln x$$

$$\Rightarrow f(x,y) = x \ln y + x \ln x = x \ln(xy)$$

$$\Rightarrow \text{Solutions to ODE: } \boxed{x \ln(xy) = C}$$

$$\textcircled{6} (2y^2x - 3)dx + (2yx^2 + 4)dy = 0$$

$$\frac{\partial M}{\partial y} = 4yx \quad \checkmark$$

$$\frac{\partial N}{\partial x} = 4yx \quad \checkmark$$

Exact

Potential: $\frac{\partial f}{\partial x} = 2y^2x - 3 \Rightarrow f(x, y) = x^2y^2 - 3x + g(y)$

$$\frac{\partial f}{\partial y} = 2yx^2 + 4$$

$$\Rightarrow \left. \begin{aligned} \frac{\partial f}{\partial y} &= 2x^2y + g'(y) \\ &= 2x^2y + 4 \end{aligned} \right\} \Rightarrow g'(y) = 4$$

$$\Rightarrow g(y) = 4y$$

$$\Rightarrow f(x, y) = x^2y^2 - 3x + 4y$$

$$\Rightarrow \text{Solution to ODE: } x^2y^2 - 3x + 4y = C$$

$$\textcircled{7} \underbrace{\left(2y - \frac{1}{x} + \cos(3x)\right)}_{N(x, y)} \frac{dy}{dx} + \underbrace{\left(\frac{y}{x^2} - 4x^3 + 3y \sin(3x)\right)}_{M(x, y)} = 0$$

$$\frac{\partial M}{\partial y} = \frac{1}{x^2} + 3 \sin(3x)$$

$$\frac{\partial N}{\partial x} = +\frac{1}{x^2} - 3 \sin(3x)$$

\Rightarrow not exact.

$$\textcircled{8} (x^3 + y^3)dx + 3xy^2dy = 0$$

$$\frac{\partial M}{\partial y} = 3y^2; \quad \frac{\partial N}{\partial x} = 3y^2 \Rightarrow \text{Exact}$$

$$\frac{\partial f}{\partial x} = x^3 + y^3 \Rightarrow f(x, y) = \frac{1}{4}x^4 + y^3x + g(y)$$

$$\Rightarrow \left. \begin{aligned} \frac{\partial f}{\partial y} &= 3y^2x + g'(y) \\ \frac{\partial f}{\partial y} &= 3xy^2 \end{aligned} \right\} \Rightarrow g'(y) = 0 \Rightarrow g(y) = C$$

$$f(x, y) = \frac{1}{4}x^4 + y^3x$$

$$\Rightarrow \text{Solution to ODE: } \frac{1}{4}x^4 + y^3x = C$$

$$(9) (y^3 - y^2 \sin x - x) dx + (3xy^2 + 2y \cos x) dy = 0$$

$$\frac{\partial M}{\partial y} = 3y^2 - 2y \sin x \quad \checkmark \quad \text{Exact}$$

$$\frac{\partial N}{\partial x} = 3y^2 - 2y \sin x \quad \checkmark$$

Potential: $\frac{\partial f}{\partial x} = y^3 - y^2 \sin x - x$

$$\Rightarrow f(x, y) = y^3 x + y^2 \cos x - \frac{x^2}{2} + g(y)$$

$$\Rightarrow \frac{\partial f}{\partial y} = 3y^2 x + 2y \cos x + g'(y) \left. \begin{array}{l} \\ \\ \end{array} \right\} \Rightarrow g'(y) = 0 \Rightarrow g(y) = C$$

$$= 3xy^2 + 2y \cos x$$

$$f(x, y) = y^3 x + y^2 \cos x - \frac{x^2}{2} \Rightarrow \text{Solution to ODE: } y^3 x + y^2 \cos x - \frac{x^2}{2} = C$$

$$(10) (y \ln y - e^{-xy}) dx + \left(\frac{1}{y} + x \ln y\right) dy = 0$$

$$\frac{\partial M}{\partial y} = \ln y + 1 + x e^{-xy} \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} \text{Not exact.}$$

$$\frac{\partial N}{\partial x} = \ln y$$

$$(11) \frac{2x}{y} dx - \frac{x^2}{y^2} dy = 0$$

$$\frac{\partial M}{\partial y} = -\frac{2x}{y^2} \quad ; \quad \frac{\partial N}{\partial x} = -\frac{2x}{y^2} \quad \Rightarrow \text{Exact}$$

$$\frac{\partial f}{\partial x} = \frac{2x}{y} \Rightarrow f(x, y) = \frac{1}{y} x^2 + g(y)$$

$$\Rightarrow \frac{\partial f}{\partial y} = -\frac{x^2}{y^2} + g'(y) \left. \begin{array}{l} \\ \\ \end{array} \right\} \Rightarrow g'(y) = 0 \Rightarrow g(y) = C$$

$$= -\frac{x^2}{y^2}$$

$$f(x, y) = \frac{1}{y} x^2 \Rightarrow \text{Solution to ODE: } \frac{1}{y} x^2 = C \quad \text{or} \quad y = C x^2$$

Remark: (#11) is also separable:

$$\frac{2x}{y} dx = \frac{x^2}{y^2} dy \Rightarrow \frac{2}{x} dx = \frac{1}{y} dy \Rightarrow 2 \ln|x| + C = \ln|y|$$

$$\Rightarrow y = \pm e^C x^2 \Rightarrow \boxed{y = cx^2} \quad (c \neq 0)$$

(12) $\left(\frac{1}{x} - \frac{y}{x^2+y^2} \right) dx + \frac{x}{x^2+y^2} dy = 0$

$$\frac{\partial M}{\partial y} = - \frac{(x^2+y^2) - 2y^2}{(x^2+y^2)^2} = - \frac{x^2 - y^2}{(x^2+y^2)^2} \quad \checkmark$$

$$\frac{\partial M}{\partial x} = \frac{(x^2+y^2) - 2x^2}{(x^2+y^2)^2} = \frac{y^2 - x^2}{(x^2+y^2)^2} \quad \checkmark$$

Exact

Potential: $\frac{\partial f}{\partial x} = \frac{1}{x} - \frac{y}{x^2+y^2}$

$$\frac{\partial f}{\partial y} = \frac{x}{x^2+y^2} \Rightarrow f(x,y) = \int \frac{x}{x^2+y^2} dy + g(x) = \arctan(y/x) + g(x)$$

$$f(x,y) = \arctan\left(\frac{y}{x}\right) + g(x)$$

$$\Rightarrow \frac{\partial f}{\partial x} = \frac{1}{1 + \left(\frac{y}{x}\right)^2} \left(-\frac{y}{x^2}\right) + g'(x)$$

$$= \frac{x^2}{x^2+y^2} \cdot \frac{-y}{x^2} + g'(x) = - \frac{y}{x^2+y^2} + g'(x)$$

$$= \frac{1}{x} - \frac{y}{x^2+y^2} \quad \left. \begin{array}{l} \Rightarrow g'(x) = \frac{1}{x} \\ \Rightarrow g(x) = \ln|x| \end{array} \right\}$$

$$\boxed{f(x,y) = \arctan(y/x) + \ln|x|}$$

\Rightarrow Solution to ODE: $\boxed{\arctan(y/x) + \ln|x| = C}$

Observation: What if, instead, we found the potential by integrating with respect to x first?

$$\frac{\partial f}{\partial x} = \frac{1}{x} - \frac{y}{x^2+y^2} \Rightarrow f(x,y) = \ln|x| - \arctan(x/y) + g(y)$$

$$\Rightarrow \frac{\partial f}{\partial y} = - \frac{1}{1+x^2/y^2} \cdot \frac{-x}{y^2} + g'(y)$$

$$= + \frac{x}{x^2+y^2} + g'(y)$$

$$= \frac{x}{x^2+y^2}$$

$$\Rightarrow f(x,y) = \ln|x| - \arctan(x/y) \Rightarrow \boxed{\ln|x| - \arctan(x/y) = C}$$

We obtained two apparently different solutions:

$$\boxed{\ln|x| + \arctan(y/x) = C} \quad (1)$$

$$\boxed{\ln|x| - \arctan(x/y) = C} \quad (2)$$

We can check by implicit differentiation that both satisfy the DE:

$$(1) \Rightarrow \frac{1}{x} + \frac{1}{1+(y/x)^2} \left(\frac{y'}{x} - \frac{y}{x^2} \right) = 0$$

$$\Rightarrow \frac{1}{x} + \frac{x^2}{x^2+y^2} \frac{xy' - y}{x^2} = 0 \Rightarrow \left(\frac{1}{x} - \frac{y}{x^2+y^2} \right) + \frac{x}{x^2+y^2} y' = 0 \quad \checkmark$$

$$(2) \Rightarrow \frac{1}{x} - \frac{1}{1+(x/y)^2} \left(\frac{1}{y} - \frac{x}{y^2} y' \right) = 0$$

$$\Rightarrow \frac{1}{x} - \frac{y^2}{x^2+y^2} \frac{y - xy'}{y^2} = 0 \Rightarrow \left(\frac{1}{x} - \frac{y}{x^2+y^2} \right) + \frac{x}{x^2+y^2} y' = 0 \quad \checkmark$$

The two solutions are equivalent, because

$$\boxed{\arctan\left(\frac{1}{\alpha}\right) = \pm \frac{\pi}{2} - \arctan(\alpha)}$$

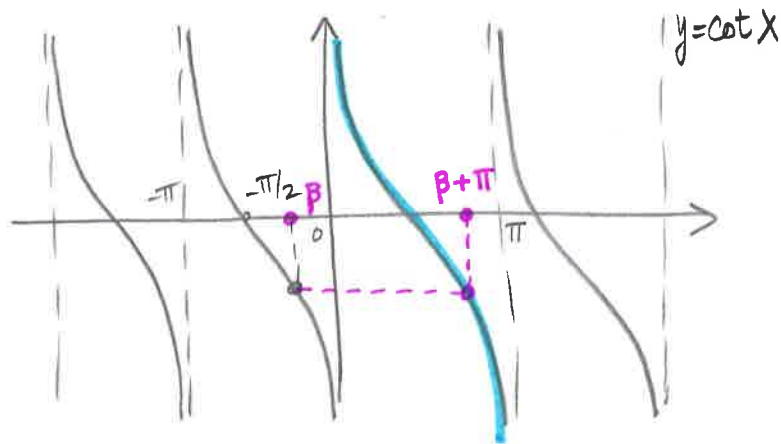
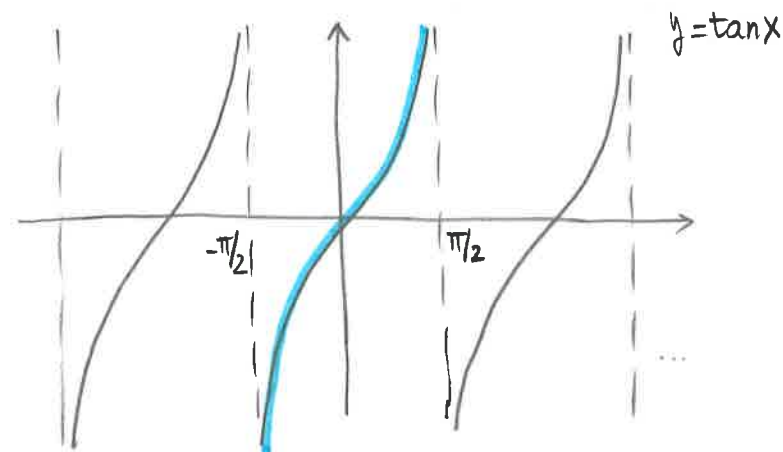
So

$$(1): \ln|x| + \arctan(y/x) = C \Leftrightarrow \ln|x| \pm \frac{\pi}{2} - \arctan(x/y) = C$$

$$\Leftrightarrow \ln|x| - \arctan(x/y) = C \quad (2)$$

→ (proof on next page, only if you care)

$$\arctan\left(\frac{1}{\alpha}\right) = \begin{cases} -\frac{\pi}{2} - \arctan(\alpha), & \text{if } \alpha < 0 \\ \frac{\pi}{2} - \arctan(\alpha), & \text{if } \alpha > 0 \end{cases}$$



Recall some basic facts from Trigonometry:

$$\arctan : \mathbb{R} \rightarrow (-\pi/2, \pi/2)$$

$$\operatorname{arccot} : \mathbb{R} \rightarrow (0, \pi)$$

$$\arctan(\alpha) + \operatorname{arccot}(\alpha) = \frac{\pi}{2}$$

Suppose $\boxed{\arctan \frac{1}{\alpha} = \beta}$
 $\rightarrow \beta \in (-\pi/2, \pi/2)$

$$\Rightarrow \frac{1}{\alpha} = \tan \beta$$

$$\Rightarrow \alpha = \frac{1}{\tan \beta}$$

$$\Rightarrow \boxed{\alpha = \cot \beta} \Rightarrow \operatorname{arccot}(\alpha) = \operatorname{arccot}(\cot \beta)$$

$$\text{If } \boxed{\alpha > 0} \Rightarrow \boxed{\beta \in (0, \pi/2)} \Rightarrow \boxed{\operatorname{arccot} \alpha = \beta}$$

$$\text{If } \boxed{\alpha < 0} \Rightarrow \boxed{\beta \in (-\pi/2, 0)} \Rightarrow \boxed{\operatorname{arccot} \alpha = \beta + \pi} \quad (\text{see pic on the right above})$$

$$\Rightarrow \beta = \arctan\left(\frac{1}{\alpha}\right) = \begin{cases} \operatorname{arccot} \alpha & \text{if } \alpha > 0 \\ \operatorname{arccot} \alpha - \pi & \text{if } \alpha < 0 \end{cases} = \begin{cases} \frac{\pi}{2} - \arctan \alpha & \text{if } \alpha > 0 \\ -\frac{\pi}{2} - \arctan \alpha & \text{if } \alpha < 0 \end{cases}$$

(13) $(x+y)^2 dx + (2xy+x^2-1) dy = 0$; $y(1)=1$

$$\frac{\partial M}{\partial y} = 2(x+y) \quad \checkmark \quad \text{Exact}$$

$$\frac{\partial N}{\partial x} = 2y+2x=2(x+y) \quad \checkmark$$

Potential: $\frac{\partial f}{\partial x} = (x+y)^2 \Rightarrow f(x,y) = \frac{1}{3}(x+y)^3 + g(y)$

$$\Rightarrow \frac{\partial f}{\partial y} = (x+y)^2 + g'(y) = 2xy+x^2-1 = (x+y)^2 - (1+y^2) \quad \left. \vphantom{\frac{\partial f}{\partial y}} \right\} \Rightarrow$$

$$\Rightarrow g'(y) = -(1+y^2) \Rightarrow g(y) = -y - \frac{1}{3}y^3$$

$$f(x,y) = \frac{1}{3}(x+y)^3 - y - \frac{1}{3}y^3$$

$$\Rightarrow \frac{1}{3}(x+y)^3 - y - \frac{1}{3}y^3 = c \quad \begin{matrix} x=1 \\ y=1 \end{matrix} \Rightarrow \frac{8}{3} - 1 - \frac{1}{3} = c \Rightarrow c = \frac{4}{3}$$

$$\Rightarrow (x+y)^3 - 3y - y^3 = 4$$

(14) $(e^x+y) dx + (2+x+ye^y) dy = 0$; $y(0)=1$

$$\frac{\partial M}{\partial y} = 1 ; \frac{\partial N}{\partial x} = 1 \quad \text{Exact}$$

$$\frac{\partial f}{\partial x} = e^x + y \Rightarrow f(x,y) = e^x + xy + g(y)$$

$$\Rightarrow \frac{\partial f}{\partial y} = x + g'(y) = 2 + x + ye^y \quad \left. \vphantom{\frac{\partial f}{\partial y}} \right\} \begin{matrix} \Rightarrow g'(y) = 2 + ye^y \\ \Rightarrow g(y) = 2y + ye^y - e^y \end{matrix}$$

$$f(x,y) = e^x + xy + 2y + ye^y - e^y$$

$$\Rightarrow e^x + xy + 2y + ye^y - e^y = c$$

$$x=0, y=1 \Rightarrow 1+2+e-e=c \Rightarrow c=3$$

$$e^x + xy + 2y + ye^y - e^y = 3$$

(15) $(y^2 \cos x - 3x^2 y - 2x) dx + (2y \sin x - x^3 + \ln y) dy = 0$; $y(0) = e$

$$\frac{\partial M}{\partial y} = 2y \cos x - 3x^2 \quad ; \quad \frac{\partial N}{\partial x} = 2y \cos x - 3x^2 \quad \underline{\text{Exact}}$$

$$\frac{\partial f}{\partial x} = y^2 \cos x - 3x^2 y - 2x \Rightarrow f(x, y) = y^2 \sin x - x^3 y - x^2 + g(y)$$

$$\Rightarrow \left. \begin{aligned} \frac{\partial f}{\partial y} &= 2y \sin x - x^3 + g'(y) \\ &= 2y \sin x - x^3 + \ln y \end{aligned} \right\} \Rightarrow$$

$$\Rightarrow g'(y) = \ln y \Rightarrow g(y) = y \ln y - y$$

$$f(x, y) = y^2 \sin x - x^3 y - x^2 + y \ln y - y \Rightarrow y^2 \sin x - x^3 y - x^2 + y \ln y - y = C$$

$$x=0, y=e \Rightarrow e - e = C \Rightarrow C = 0$$

$$\Rightarrow y^2 \sin x - x^3 y - x^2 + y \ln y - y = 0$$

(16) $\left(\frac{1}{1+y^2} + \cos x - 2xy \right) \frac{dy}{dx} = y(y + \sin x)$; $y(0) = 1$

$$y(y + \sin x) dx - \left(\frac{1}{1+y^2} + \cos x - 2xy \right) dy = 0$$

$$M(x, y) = y(y + \sin x) = y^2 + y \sin x \Rightarrow \frac{\partial M}{\partial y} = 2y + \sin x \quad \checkmark \quad \underline{\text{Exact}}$$

$$N(x, y) = -\left(\frac{1}{1+y^2} + \cos x - 2xy \right) \Rightarrow \frac{\partial N}{\partial x} = \sin x + 2y \quad \checkmark$$

$$\frac{\partial f}{\partial x} = y^2 + y \sin x \Rightarrow f(x, y) = xy^2 - y \cos x + g(y)$$

$$\Rightarrow \left. \begin{aligned} \frac{\partial f}{\partial y} &= 2xy - \cos x + g'(y) \\ &= 2xy - \cos x - \frac{1}{1+y^2} \end{aligned} \right\} \Rightarrow g'(y) = -\frac{1}{1+y^2}$$

$$\Rightarrow g(y) = -\arctan(y)$$

$$\Rightarrow f(x, y) = xy^2 - y \cos x - \arctan(y) = C$$

$$x=0, y=1 \Rightarrow -1 - \frac{\pi}{4} = C \Rightarrow xy^2 - y \cos x - \arctan(y) = -1 - \frac{\pi}{4}$$

$$(17) (y^3 + kxy^4 - 2x) dx + (3xy^2 + 20x^2y^3) dy = 0$$

$$\left. \begin{aligned} M(x,y) = y^3 + kxy^4 - 2x &\Rightarrow \frac{\partial M}{\partial y} = 3y^2 + 4kxy^3 \\ N(x,y) = 3xy^2 + 20x^2y^3 &\Rightarrow \frac{\partial N}{\partial x} = 3y^2 + 40xy^3 \end{aligned} \right\} \Rightarrow 4k = 40 \Rightarrow k = 10$$

Solve: $\frac{\partial f}{\partial x} = y^3 + 10xy^4 - 2x \Rightarrow f(x,y) = xy^3 + 5x^2y^4 - x^2 + g(y)$

$$\Rightarrow \left. \begin{aligned} \frac{\partial f}{\partial y} &= 3xy^2 + 20x^2y^3 + g'(y) \\ &= 3xy^2 + 20x^2y^3 \end{aligned} \right\} \Rightarrow g'(y) = 0 \\ \Rightarrow g(y) = c$$

Solution: $xy^3 + 5x^2y^4 - x^2 = c$

$$(18) (2x - y \sin(xy) + ky^4) dx - (20xy^3 + x \sin(xy)) dy = 0$$

$$\left. \begin{aligned} \frac{\partial M}{\partial y} &= -\sin(xy) - xy \cos(xy) + 4ky^3 \\ \frac{\partial N}{\partial x} &= -20y^3 - \sin(xy) - xy \cos(xy) \end{aligned} \right\} \Rightarrow 4k = -20 \Rightarrow k = -5$$

Solve: $\frac{\partial f}{\partial x} = 2x - y \sin(xy) - 5y^4 \Rightarrow f(x,y) = x^2 + \cos(xy) - 5y^4x + g(y)$

$$\Rightarrow \left. \begin{aligned} \frac{\partial f}{\partial y} &= -x \sin(xy) - 20y^3x + g'(y) \\ &= -x \sin(xy) - 20y^3x \end{aligned} \right\} \Rightarrow g'(y) = 0 \\ g(y) = c$$

Solution: $x^2 + \cos(xy) - 5y^4x = c$

$$(19) (2xy^2 + ye^x) dx + (2x^2y + ke^x - 1) dy = 0$$

$$\left. \begin{aligned} \frac{\partial M}{\partial y} &= 4xy + e^x \\ \frac{\partial N}{\partial x} &= 4xy + ke^x \end{aligned} \right\} \Rightarrow \boxed{k=1}$$

Solve: $\frac{\partial f}{\partial x} = 2xy^2 + ye^x \Rightarrow f(x, y) = x^2y^2 + ye^x + g(y)$

$$\Rightarrow \left. \begin{aligned} \frac{\partial f}{\partial y} &= 2x^2y + e^x + g'(y) \\ &= 2x^2y + e^x - 1 \end{aligned} \right\} \Rightarrow \begin{aligned} g'(y) &= -1 \\ g(y) &= -y \end{aligned}$$

\Rightarrow Solution: $\boxed{x^2y^2 + ye^x - y = c}$

$$(20) (6xy^3 + \cos y) dx + (kx^2y^2 - x \sin y) dy = 0$$

$$\left. \begin{aligned} \frac{\partial M}{\partial y} &= 18xy^2 - \sin y \\ \frac{\partial N}{\partial x} &= 2kxy^2 - \sin y \end{aligned} \right\} \Rightarrow 18 = 2k \Rightarrow \boxed{k=9}$$

Solve: $\frac{\partial f}{\partial x} = 6xy^3 + \cos y \Rightarrow f(x, y) = 3x^2y^3 + x \cos y + g(y)$

$$\Rightarrow \left. \begin{aligned} \frac{\partial f}{\partial y} &= 9x^2y^2 - x \sin y + g'(y) \\ &= 9x^2y^2 - x \sin y \end{aligned} \right\} \Rightarrow \begin{aligned} g'(y) &= 0 \\ g(y) &= c \end{aligned}$$

\Rightarrow Solution: $\boxed{3x^2y^3 + x \cos y = c}$

$$(21) \quad M(x,y) dx + \left(x e^{xy} + 2xy + \frac{1}{x} \right) dy = 0$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow \frac{\partial M}{\partial y} = e^{xy} + xy e^{xy} + 2y - \frac{1}{x^2}$$

$$\Rightarrow \frac{\partial M}{\partial y} = e^{xy}(1+xy) + 2y - \frac{1}{x^2}$$

$$\Rightarrow M(x,y) = y e^{xy} + y^2 - \frac{y}{x^2} + g(x)$$

$$\int e^{xy}(1+xy) dy = \frac{1}{x} \int (1+xy) (e^{xy})' dy$$

$$= \frac{1}{x} (1+xy) e^{xy} - \frac{1}{x} \int x e^{xy} dy$$

$$= \left(\frac{1}{x} + y \right) e^{xy} - \frac{1}{x} e^{xy}$$

$$= y e^{xy}$$

→ You can choose any function of x you want here

$$(22) \quad \left(\frac{\sqrt{y}}{\sqrt{x}} + \frac{x}{x^2+y} \right) dx + N(x,y) dy = 0$$

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y} \Rightarrow \frac{\partial N}{\partial x} = \frac{1}{2\sqrt{xy}} - \frac{x}{(x^2+y)^2}$$

$$\Rightarrow N(x,y) = \frac{\sqrt{x}}{\sqrt{y}} + \frac{1}{2(x^2+y)} + g(y)$$

→ this can be any function of y

$$(23) \quad \overbrace{y(x+y+1)}^{M_1(x,y)} dx + \overbrace{(x+2y)}^{N_1(x,y)} dy = 0 ; \quad \mu(x,y) = e^x$$

$$(*) \quad ye^x(x+y+1)dx + (xe^x + 2ye^x)dy = 0$$

$$\frac{\partial M}{\partial y} = e^x(x+2y+1) \quad \checkmark$$

$$\frac{\partial N}{\partial x} = e^x + xe^x + 2ye^x = e^x(x+2y+1) \quad \checkmark$$

Exact

Remark: The original equation is not exact: $\frac{\partial M_1}{\partial y} = x+2y+1$

$$\frac{\partial N_1}{\partial x} = 1$$

Solve (*): $\frac{\partial f}{\partial x} = xy e^x + y^2 e^x + ye^x$

$$\frac{\partial f}{\partial y} = xe^x + 2ye^x \Rightarrow f(x,y) = xy e^x + y^2 e^x + g(x)$$

$$\Rightarrow \frac{\partial f}{\partial x} = ye^x + xy e^x + y^2 e^x + g'(x) \Rightarrow g'(x) = 0$$

Solution: $\boxed{xy e^x + y^2 e^x = C}$

$$(24) \quad (-xy \sin x + 2y \cos x) dx + 2x \cos x dy = 0 ; \quad \mu(x,y) = xy$$

$$(*) \quad (-x^2 y^2 \sin x + 2xy^2 \cos x) dx + 2x^2 y \cos x dy = 0$$

$$\frac{\partial M}{\partial y} = -2x^2 y \sin x + 4xy \cos x ; \quad \frac{\partial N}{\partial x} = 4xy \cos x - 2x^2 y \sin x \Rightarrow \underline{\underline{Exact}}$$

Solve: $\frac{\partial f}{\partial y} = 2x^2 y \cos x \Rightarrow f(x,y) = x^2 y^2 \cos x + g(x)$

$$\Rightarrow \left. \begin{aligned} \frac{\partial f}{\partial x} &= 2xy^2 \cos x - x^2 y^2 \sin x + g'(x) \\ &= 2xy^2 \cos x - x^2 y^2 \sin x \end{aligned} \right\} \Rightarrow g'(x) = 0 \\ g(x) = C$$

Solution: $\boxed{x^2 y^2 \cos x = C}$