① $\vec{x}' = \begin{pmatrix} 3 & -3 \\ 2 & -2 \end{pmatrix}\vec{x} + \begin{pmatrix} 4 \\ -1 \end{pmatrix}$

$$\begin{vmatrix} 3-\lambda & -3 \\ 2 & -2-\lambda \end{vmatrix} = (\lambda+2)(\lambda-3)+6 = \lambda^2-\lambda = \lambda(\lambda-1)$$

$\lambda_1=0$ $\quad \left(\begin{array}{cc|c} 3 & -3 & 0 \\ 2 & -2 & 0 \end{array}\right) \Rightarrow v_1 = v_2 \Rightarrow \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda_1=1$ $\quad \left(\begin{array}{cc|c} 2 & -3 & 0 \\ 2 & -3 & 0 \end{array}\right) \Rightarrow v_1 = \frac{3}{2}v_2 \Rightarrow \vec{v}_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

$$\boxed{\Phi(t) = \begin{pmatrix} 1 & 3e^t \\ 1 & 2e^t \end{pmatrix}} \quad \boxed{\vec{x}_c = \Phi(t)\vec{c}}$$

$$\Phi^{-1}(t) = \frac{1}{-e^t}\begin{pmatrix} 2e^t & -3e^t \\ -1 & 1 \end{pmatrix} \Rightarrow \boxed{\Phi^{-1}(t) = \begin{pmatrix} -2 & 3 \\ e^{-t} & -e^{-t} \end{pmatrix}}$$

$$\Rightarrow \Phi^{-1}(t)\vec{g}(t) = \begin{pmatrix} -11 \\ 5e^{-t} \end{pmatrix}$$

$$\Rightarrow \vec{x}_p = \begin{pmatrix} 1 & 3e^t \\ 1 & 2e^t \end{pmatrix}\begin{pmatrix} -11t \\ -5e^{-t} \end{pmatrix} = \begin{pmatrix} -11t-15 \\ -11t-10 \end{pmatrix} \Rightarrow \boxed{\vec{x}_p = -11t\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 15 \\ 10 \end{pmatrix}}$$

② $\vec{x}' = \begin{pmatrix} 3 & 2 \\ -2 & -1 \end{pmatrix}\vec{x} + \begin{pmatrix} 2e^{-t} \\ e^{-t} \end{pmatrix}$ $\quad \begin{vmatrix} 3-\lambda & 2 \\ -2 & -1-\lambda \end{vmatrix} = (\lambda+1)(\lambda-3)+4 = \lambda^2-2\lambda+1 = (\lambda-1)^2$

$\lambda_1 = \lambda_2 = 1$

$$\left(\begin{array}{cc|c} 2 & 2 & 0 \\ -2 & -2 & 0 \end{array}\right) \Rightarrow v_1 = -v_2 \Rightarrow \boxed{\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}} \Rightarrow \boxed{\vec{x}_1 = e^t\begin{pmatrix} 1 \\ -1 \end{pmatrix}}$$

$$\begin{pmatrix} 2 & 2 \\ -2 & -2 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow w_1 + w_2 = 1/2 \Rightarrow \vec{w} = \begin{pmatrix} 0 \\ 1/2 \end{pmatrix} \Rightarrow \boxed{\vec{x}_2 = te^t\begin{pmatrix} 1 \\ -1 \end{pmatrix} + e^t\begin{pmatrix} 0 \\ 1/2 \end{pmatrix}}$$

$$\Phi(t) = \begin{pmatrix} e^t & te^t \\ -e^t & -te^t + 1/2 e^t \end{pmatrix} = e^t\begin{pmatrix} 1 & t \\ -1 & -t+1/2 \end{pmatrix} \qquad \vec{x}_c = \Phi(t)\vec{c}$$

$$\Rightarrow \Phi^{-1}(t) = \frac{2}{e^{2t}}\begin{pmatrix} -te^t + 1/2 e^t & -te^t \\ +e^t & e^t \end{pmatrix} = 2e^{-t}\begin{pmatrix} -t+1/2 & -t \\ 1 & 1 \end{pmatrix}$$

$$\boxed{\int t e^{at} dt = \frac{1}{a} t e^{at} - \frac{1}{a^2} e^{at}}$$

$$\Rightarrow \Phi^{-1}(t)\vec{g}(t) = 2e^{-2t}\begin{pmatrix} -3t+1 \\ 3 \end{pmatrix}$$

$$\Rightarrow \int \Phi^{-1}(t)\vec{g}(t)\,dt = 2\int \begin{pmatrix} -3te^{-2t}+e^{-2t} \\ 3e^{-2t} \end{pmatrix} dt = 2\begin{pmatrix} \frac{3}{2}te^{-2t}+\frac{1}{4}e^{-2t} \\ -\frac{3}{2}e^{-2t} \end{pmatrix}$$

$$= e^{-2t}\begin{pmatrix} 3t+\frac{1}{2} \\ -3 \end{pmatrix}$$

$$\Rightarrow \vec{x}_p = e^{-t}\begin{pmatrix} 3t+\frac{1}{2}-3t \\ -3t-\frac{1}{2}+3t-\frac{3}{2} \end{pmatrix} = e^{-t}\begin{pmatrix} \frac{1}{2} \\ -2 \end{pmatrix} \qquad \boxed{\vec{x}_p = e^{-t}\begin{pmatrix} \frac{1}{2} \\ -2 \end{pmatrix}}$$

③ $\vec{x}' = \begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix}\vec{x} + \begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{t}$

$$\begin{vmatrix} -\lambda & 2 \\ -1 & 3-\lambda \end{vmatrix} = \lambda(\lambda-3)+2 = \lambda^2-3\lambda+2 = (\lambda-1)(\lambda-2)$$

$\lambda_1=1$: $\left(\begin{array}{cc|c} -1 & 2 & 0 \\ -1 & 2 & 0 \end{array}\right) \Rightarrow v_1 = 2v_2 \Rightarrow \vec{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\lambda_2=2$: $\left(\begin{array}{cc|c} -2 & 2 & 0 \\ -1 & 1 & 0 \end{array}\right) \Rightarrow v_1 = v_2 \Rightarrow \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\boxed{\Phi(t) = \begin{pmatrix} 2e^{t} & e^{2t} \\ e^{t} & e^{2t} \end{pmatrix}} \quad \vec{x}_c = \Phi(t)\vec{c}$$

$$\Phi^{-1}(t) = \frac{1}{e^{3t}}\begin{pmatrix} e^{2t} & -e^{2t} \\ -e^{t} & 2e^{t} \end{pmatrix} = \begin{pmatrix} e^{-t} & -e^{-t} \\ -e^{-2t} & 2e^{-2t} \end{pmatrix} \Rightarrow \Phi^{-1}(t)\vec{g}(t) = e^{t}\begin{pmatrix} 2e^{-t} \\ -3e^{-2t} \end{pmatrix} = \begin{pmatrix} 2 \\ -3e^{-t} \end{pmatrix}$$

$$\Rightarrow \vec{x}_p = \Phi(t)\begin{pmatrix} 2t \\ +3e^{-t} \end{pmatrix} = \begin{pmatrix} 4te^{t}+3e^{t} \\ 2te^{t}+3e^{t} \end{pmatrix} \qquad \boxed{\vec{x}_p = 3e^{t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 2te^{t}\begin{pmatrix} 2 \\ 1 \end{pmatrix}}$$

④ $\vec{x}' = \begin{pmatrix} 3 & -5 \\ 3/4 & -1 \end{pmatrix} \vec{x} + e^{t/2} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$\begin{vmatrix} 3-\lambda & -5 \\ 3/4 & -1-\lambda \end{vmatrix} = (\lambda+1)(\lambda-3) + \frac{15}{4} = \lambda^2 - 2\lambda + \frac{3}{4} = (\lambda-1)^2 - \frac{1}{4} = (\lambda - 3/2)(\lambda - 1/2)$$

$$\lambda_1 = 3/2 : \left(\begin{array}{cc|c} 3/2 & -5 & 0 \\ 3/4 & -5/2 & 0 \end{array}\right) \Rightarrow \frac{3}{2} v_1 = 5 v_2 \Rightarrow v_1 = \frac{10}{3} v_2 \Rightarrow \vec{v}_1 = \begin{pmatrix} 10 \\ 3 \end{pmatrix}$$

$$\lambda_2 = 1/2 : \left(\begin{array}{cc|c} 5/2 & -5 & 0 \\ 3/4 & -3/2 & 0 \end{array}\right) \Rightarrow v_1 = 2 v_2 \Rightarrow \vec{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\Rightarrow \boxed{\Phi(t) = \begin{pmatrix} 10 e^{3/2 t} & 2 e^{1/2 t} \\ 3 e^{3/2 t} & e^{1/2 t} \end{pmatrix}}$$

$$\Rightarrow \Phi^{-1}(t) = \frac{1}{4e^{2t}} \begin{pmatrix} e^{1/2 t} & -2 e^{1/2 t} \\ -3 e^{3/2 t} & 10 e^{3/2 t} \end{pmatrix} \Rightarrow \Phi^{-1}(t) \vec{g}(t) = \frac{1}{4e^{2t}} \begin{pmatrix} 3e^{t} \\ -13 e^{2t} \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 3e^{-t} \\ -13 \end{pmatrix}$$

$$\Rightarrow \vec{x}_p = \Phi(t) \frac{1}{4} \begin{pmatrix} -3e^{-t} \\ -13t \end{pmatrix} = -\frac{1}{4} \begin{pmatrix} 10 e^{3/2 t} & 2 e^{1/2 t} \\ 3 e^{3/2 t} & e^{1/2 t} \end{pmatrix} \begin{pmatrix} 3e^{-t} \\ 13t \end{pmatrix} = -\frac{1}{4} \begin{pmatrix} 30 e^{1/2 t} + 26 t e^{1/2 t} \\ 9 e^{1/2 t} + 13 t e^{1/2 t} \end{pmatrix}$$

$$\Rightarrow \boxed{\vec{x}_p = -\frac{3}{4} e^{1/2 t} \begin{pmatrix} 10 \\ 3 \end{pmatrix} - \frac{13}{4} t e^{1/2 t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}}$$

⑤ $\vec{x}' = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \vec{x} + \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} e^{t}$

$$\begin{vmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda + 1 + 1 = \lambda^2 - 2\lambda + 2 \qquad \lambda = \frac{2 \pm 2i}{2} = 1 \pm i$$

$$\Delta = 4 - 8 = -4$$

$$\lambda = 1 + i \qquad \left(\begin{array}{cc|c} -i & -1 & 0 \\ 1 & -i & 0 \end{array}\right) \Rightarrow v_1 = i v_2 \Rightarrow v = \begin{pmatrix} 1 \\ -i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$\vec{x}_1 = e^{t} \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos(t) - \begin{pmatrix} 0 \\ -1 \end{pmatrix} \sin(t) \right) = e^{t} \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$$

$$\vec{x}_2 = e^{t} \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin(t) + \begin{pmatrix} 0 \\ -1 \end{pmatrix} \cos(t) \right) = e^{t} \begin{pmatrix} \sin t \\ -\cos t \end{pmatrix}$$

$$\Rightarrow \Phi(t) = e^t \begin{pmatrix} \cos t & \sin t \\ \sin t & -\cos t \end{pmatrix} \Rightarrow \Phi^{-1}(t) = -\frac{1}{e^t} \begin{pmatrix} -\cos t & -\sin t \\ -\sin t & \cos t \end{pmatrix}$$

$$= e^{-t} \begin{pmatrix} \cos t & \sin t \\ \sin t & -\cos t \end{pmatrix}$$

$$\Rightarrow \Phi^{-1}(t)\vec{g}(t) = \begin{pmatrix} \cos t & \sin t \\ \sin t & -\cos t \end{pmatrix}\begin{pmatrix} \cos t \\ \sin t \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\Rightarrow \vec{x}_p = e^t \begin{pmatrix} \cos t & \sin t \\ \sin t & -\cos t \end{pmatrix}\begin{pmatrix} t \\ 0 \end{pmatrix} = te^t \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} \Rightarrow \boxed{\vec{x}_p = te^t \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}}$$

⑥ $\vec{x}' = \begin{pmatrix} 1 & 8 \\ 1 & -1 \end{pmatrix}\vec{x} + \begin{pmatrix} 12 \\ 12 \end{pmatrix} t$

$$\begin{vmatrix} 1-\lambda & 8 \\ 1 & -1-\lambda \end{vmatrix} = (\lambda+1)(\lambda-1) - 8 = \lambda^2 - 9 = (\lambda-3)(\lambda+3)$$

$$\lambda_1 = 3: \quad \left(\begin{array}{cc|c} -2 & 8 & 0 \\ 1 & -4 & 0 \end{array}\right) \Rightarrow v_1 = 4v_2 \Rightarrow \vec{v}_1 = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

$$\lambda_2 = -3: \quad \left(\begin{array}{cc|c} +4 & 8 & 0 \\ 1 & 2 & 0 \end{array}\right) \Rightarrow v_1 = -2v_2 \Rightarrow \vec{v}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$$\boxed{\Phi(t) = \begin{pmatrix} 4e^{3t} & -2e^{-3t} \\ e^{3t} & e^{-3t} \end{pmatrix}}$$

$$\Phi^{-1}(t) = \frac{1}{6}\begin{pmatrix} e^{-3t} & 2e^{-3t} \\ -e^{+3t} & 4e^{3t} \end{pmatrix} \Rightarrow \Phi^{-1}(t)\vec{g}(t) = 2t\begin{pmatrix} 3e^{-3t} \\ 3e^{3t} \end{pmatrix} = 6\begin{pmatrix} te^{-3t} \\ te^{3t} \end{pmatrix}$$

$$\Rightarrow \vec{x}_p = \begin{pmatrix} 4e^{3t} & -2e^{-3t} \\ e^{3t} & e^{-3t} \end{pmatrix} 6 \begin{pmatrix} -1/3\, te^{-3t} - 1/9\, e^{-3t} \\ 1/3\, te^{3t} - 1/9\, e^{3t} \end{pmatrix} = -\frac{2}{3}\begin{pmatrix} 4e^{3t} & -2e^{-3t} \\ e^{3t} & e^{-3t} \end{pmatrix}\begin{pmatrix} 3te^{-3t} + e^{-3t} \\ -3te^{3t} + e^{3t} \end{pmatrix}$$

$$= -\frac{2}{3}\begin{pmatrix} 18t + 2 \\ 2 \end{pmatrix} = -12\begin{pmatrix} t \\ 0 \end{pmatrix} - \frac{4}{3}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \boxed{\vec{x}_p = -12\begin{pmatrix} t \\ 0 \end{pmatrix} - \frac{4}{3}\begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

7. $$\vec{x}' = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \vec{x} + \begin{pmatrix} e^t \\ e^{2t} \\ e^{3t} \cdot t \end{pmatrix}$$

$$\begin{vmatrix} 1-\lambda & 1 & 0 \\ 1 & 1-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} = (3-\lambda)\left(\lambda^2-2\lambda+1-1\right) = \lambda(\lambda-2)(3-\lambda)$$

$\lambda=0$: $\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{array}\right)$ $\Rightarrow v_1 = -v_2$, $\Rightarrow v_3 = 0$ $\Rightarrow \vec{v_1} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

$\lambda=2$: $\left(\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$ $\Rightarrow v_1 = v_2$, $\Rightarrow v_3 = 0$ $\Rightarrow \vec{v_2} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

$\lambda=3$: $\left(\begin{array}{ccc|c} -2 & 1 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$ $\left.\begin{array}{l} \Rightarrow v_2 = 2v_1 \\ \Rightarrow v_1 = 2v_2 \end{array}\right\} \Rightarrow v_1 = v_2 = 0$

$\Rightarrow \vec{v_3} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

$$\Phi(t) = \begin{pmatrix} 1 & e^{2t} & 0 \\ -1 & e^{2t} & 0 \\ 0 & 0 & e^{3t} \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & e^{2t} & 0 & 1 & 0 & 0 \\ -1 & e^{2t} & 0 & 0 & 1 & 0 \\ 0 & 0 & e^{3t} & 0 & 0 & 1 \end{array}\right) \xrightarrow[R_3/e^{3t}]{R_2+R_1} \left(\begin{array}{ccc|ccc} 1 & e^{2t} & 0 & 1 & 0 & 0 \\ 0 & 2e^{2t} & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & e^{-3t} \end{array}\right)$$

$$\xrightarrow{R_1-\frac{1}{2}R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & -1/2 & 0 \\ 0 & 2e^{2t} & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & e^{-3t} \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & -1/2 & 0 \\ 0 & 1 & 0 & 1/2 e^{-2t} & 1/2 e^{-2t} & 0 \\ 0 & 0 & 1 & 0 & 0 & e^{-3t} \end{array}\right)$$

$$\Rightarrow \Phi^{-1}(t) = \begin{pmatrix} 1/2 & -1/2 & 0 \\ 1/2 e^{-2t} & 1/2 e^{-2t} & 0 \\ 0 & 0 & e^{-3t} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -1 & 0 \\ e^{-2t} & e^{-2t} & 0 \\ 0 & 0 & 2e^{-3t} \end{pmatrix}$$

$$\Rightarrow \Phi^{-1}(t)\,\vec{g}(t) = \frac{1}{2}\begin{pmatrix} 1 & -1 & 0 \\ e^{-2t} & e^{-2t} & 0 \\ 0 & 0 & 2e^{-3t} \end{pmatrix}\begin{pmatrix} e^{t} \\ e^{2t} \\ te^{3t} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} e^{t}-e^{2t} \\ e^{-t}+1 \\ 2t \end{pmatrix}$$

$$\Rightarrow \vec{X}_p = \frac{1}{2}\begin{pmatrix} 1 & e^{2t} & 0 \\ -1 & e^{2t} & 0 \\ 0 & 0 & e^{3t} \end{pmatrix}\begin{pmatrix} e^{t}-\frac{1}{2}e^{2t} \\ -e^{-t}+t \\ t^2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} e^{t}-\frac{1}{2}e^{2t}-e^{t}+te^{2t} \\ -e^{t}+\frac{1}{2}e^{2t}-e^{t}+te^{2t} \\ t^2e^{3t} \end{pmatrix}$$

$$\Rightarrow \boxed{\vec{X}_p = \begin{pmatrix} -\frac{1}{4}e^{2t}+\frac{1}{2}te^{2t} \\ -e^{t}+\frac{1}{4}e^{2t}+\frac{1}{2}te^{2t} \\ \frac{1}{2}t^2e^{3t} \end{pmatrix}}$$

8 $A = \begin{pmatrix} 3 & -3 \\ 2 & -2 \end{pmatrix}$; $\Phi(t) = \begin{pmatrix} 1 & 3e^{t} \\ 1 & 2e^{t} \end{pmatrix}$; $\Phi^{-1}(t) = \begin{pmatrix} -2 & 3 \\ e^{-t} & -e^{-t} \end{pmatrix}$

$$e^{tA} = \Phi(t)\,\Phi^{-1}(0) = \begin{pmatrix} 1 & 3e^{t} \\ 1 & 2e^{t} \end{pmatrix}\begin{pmatrix} -2 & 3 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} -2+3e^{t} & 3-3e^{t} \\ -2+2e^{t} & 3-2e^{t} \end{pmatrix}$$

9 $A = \begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix}$; $\Phi(t) = \begin{pmatrix} 2e^{t} & e^{2t} \\ e^{t} & e^{2t} \end{pmatrix}$; $\Phi^{-1}(t) = \begin{pmatrix} e^{-t} & -e^{-t} \\ -e^{-2t} & 2e^{-2t} \end{pmatrix}$

$$e^{tA} = \Phi(t)\,\Phi^{-1}(0) = \begin{pmatrix} 2e^{t} & e^{2t} \\ e^{t} & e^{2t} \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 2e^{t}-e^{2t} & -2e^{t}+2e^{2t} \\ e^{t}-e^{2t} & -e^{t}+2e^{2t} \end{pmatrix}$$

(10) $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$; $\Phi(t) = \begin{pmatrix} 1 & e^{2t} & 0 \\ -1 & e^{2t} & 0 \\ 0 & 0 & e^{3t} \end{pmatrix}$

Find e^{tA} without inverting $\Phi(t)$:

General Solution to $\vec{x}' = A\vec{x}$: $\vec{x} = \Phi(t)\vec{c} = \begin{pmatrix} c_1 + c_2 e^{2t} \\ -c_1 + c_2 e^{2t} \\ c_3 e^{3t} \end{pmatrix} \Rightarrow \vec{x}(0) = \begin{pmatrix} c_1 + c_2 \\ -c_1 + c_2 \\ c_3 \end{pmatrix}$

$$\vec{x}(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} c_1 + c_2 = 1 \\ -c_1 + c_2 = 0 \\ c_3 = 0 \end{matrix} \Rightarrow \vec{c} = \begin{pmatrix} 1/2 \\ 1/2 \\ 0 \end{pmatrix} \Rightarrow \vec{x}_1 = \begin{pmatrix} 1/2 + 1/2\, e^{2t} \\ -1/2 + 1/2\, e^{2t} \\ 0 \end{pmatrix}$$

$$\vec{x}(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} c_1 + c_2 = 0 \\ -c_1 + c_2 = 1 \\ c_3 = 0 \end{matrix} \Rightarrow \vec{c} = \begin{pmatrix} -1/2 \\ 1/2 \\ 0 \end{pmatrix} \Rightarrow \vec{x}_2 = \begin{pmatrix} -1/2 + 1/2\, e^{2t} \\ 1/2 + 1/2\, e^{2t} \\ 0 \end{pmatrix}$$

$$\vec{x}(0) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \Rightarrow \begin{matrix} c_1 + c_2 = 0 \\ -c_1 + c_2 = 0 \\ c_3 = 1 \end{matrix} \Rightarrow \vec{c} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \Rightarrow \vec{x}_3 = \begin{pmatrix} 0 \\ 0 \\ e^{3t} \end{pmatrix}$$

$$\Rightarrow e^{tA} = \begin{pmatrix} 1/2 + 1/2\, e^{2t} & -1/2 + 1/2\, e^{2t} & 0 \\ -1/2 + 1/2\, e^{2t} & 1/2 + 1/2\, e^{2t} & 0 \\ 0 & 0 & e^{3t} \end{pmatrix}$$

(11)

$$e^{tA} = \begin{pmatrix} 5e^{t} - e^{2t} - 3e^{-t} & 3e^{t} - e^{2t} - 2e^{-t} & -e^{t} + e^{-t} \\ -5e^{t} + 2e^{2t} + 3e^{-t} & -3e^{t} + 2e^{2t} + 2e^{-t} & e^{t} - e^{-t} \\ 5e^{t} + e^{2t} - 6e^{-t} & 3e^{t} + e^{2t} - 4e^{-t} & -e^{t} + 2e^{-t} \end{pmatrix}$$

(a).

$\Rightarrow$ General solution to $\vec{x}' = A\vec{x}$: $\quad \vec{x} = e^{tA}\vec{c}$

$$\vec{x} = e^{tA}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \Rightarrow \vec{x}(0) = e^{0}\vec{c} = I\vec{c} \Rightarrow \vec{x}(0) = \vec{c} \Rightarrow \vec{c} = \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix}$$

$\Rightarrow$ Solution to IVP:

$$\vec{x} = e^{tA}\begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} -10e^{t} + 2e^{2t} + 6e^{-t} + 3e^{t} - e^{2t} - 2e^{-t} - 4e^{t} + 4e^{-t} \\ 10e^{t} - 4e^{2t} - 6e^{-t} - 3e^{t} + 2e^{2t} + 2e^{-t} + 4e^{t} - 4e^{-t} \\ -10e^{t} - 2e^{2t} + 12e^{-t} + 3e^{t} + e^{2t} - 4e^{-t} - 4e^{t} + 8e^{-t} \end{pmatrix}$$

$$\Rightarrow \boxed{\vec{x} = \begin{pmatrix} -11e^{t} + e^{2t} + 8e^{-t} \\ 11e^{t} - 2e^{2t} - 8e^{-t} \\ -11e^{t} - e^{2t} + 16e^{-t} \end{pmatrix}}$$

(b). Particular solution to $\vec{x}' = A\vec{x} + e^{t}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$.

$$e^{-tA}\vec{g}(t) = \begin{pmatrix} 5e^{-t} - e^{-2t} - 3e^{t} & 3e^{-t} - e^{-2t} - 2e^{t} & -e^{-t} + e^{t} \\ -5e^{-t} + 2e^{-2t} + 3e^{t} & -3e^{-t} + 2e^{-2t} + 2e^{t} & e^{-t} - e^{t} \\ 5e^{-t} + e^{-2t} - 6e^{t} & 3e^{-t} + e^{-2t} - 4e^{t} & -e^{-t} + 2e^{t} \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}e^{t} =$$

$$= \begin{pmatrix} e^{-t} \\ -e^{-t} \\ e^{-t} \end{pmatrix} e^{t} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \quad \Rightarrow \quad \int e^{-tA} \vec{g}(t)\,dt = \begin{pmatrix} t \\ -t \\ t \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} t$$

$$\Rightarrow \vec{X}_p = e^{tA} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} t = \begin{pmatrix} e^{t} \\ -e^{t} \\ e^{t} \end{pmatrix} t = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} te^{t}$$

$$\boxed{\vec{X}_p = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} te^{t}}$$