Math 2552 - Differential Equations **Homework 21**
Sections F1 – F4; L1 – L4
Georgia Institute of Technology, Fall 2015

Non-Homogeneous Linear Systems with Constant Coefficients: Variation of Parameters; Complex Exponential

Use variation of parameters to find a particular solution to the systems:

1. $\mathbf{x}' = \begin{pmatrix} 3 & -3 \\ 2 & -2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 4 \\ -1 \end{pmatrix}$

2. $\mathbf{x}' = \begin{pmatrix} 3 & 2 \\ -2 & -1 \end{pmatrix}\mathbf{x} + e^{-t}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

3. $\mathbf{x}' = \begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix}\mathbf{x} + e^{t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

4. $\mathbf{x}' = \begin{pmatrix} 3 & -5 \\ 3/4 & -1 \end{pmatrix}\mathbf{x} + e^{t/2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

5. $\mathbf{x}' = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\mathbf{x} + e^{t}\begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$

6. $\mathbf{x}' = \begin{pmatrix} 1 & 8 \\ 1 & -1 \end{pmatrix}\mathbf{x} + t\begin{pmatrix} 12 \\ 12 \end{pmatrix}$

7. $\mathbf{x}' = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}\mathbf{x} + \begin{pmatrix} e^t \\ e^{2t} \\ te^{3t} \end{pmatrix}$

8. Use your results in **1.** to find e^{tA} for

$$A = \begin{pmatrix} 3 & -3 \\ 2 & -2 \end{pmatrix}.$$

9. Use your results in **3.** to find e^{tA} for

$$A = \begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix}.$$

10. Use your results in **7.** to find e^{tA} for

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

11. Suppose that $A \in \mathbb{R}^{3\times 3}$ has exponential

$$e^{tA} = \begin{pmatrix} 5e^t - e^{2t} - 3e^{-t} & 3e^t - e^{2t} - 2e^{-t} & -e^t + e^{-t} \\ -5e^t + 2e^{2t} + 3e^{-t} & -3e^t + 2e^{2t} + 2e^{-t} & e^t - e^{-t} \\ 5e^t + e^{2t} - 6e^{-t} & 3e^t + e^{2t} - 4e^{-t} & -e^t + 2e^{-t} \end{pmatrix}$$

a). Find the solution to the initial value problem:

$$\mathbf{x}' = A\mathbf{x};\ \ \mathbf{x}(0) = \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix}.$$

b). Find a particular solution to:

$$\mathbf{x}' = A\mathbf{x} + e^t\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

Non-Homogeneous Linear Systems : Variation of Parameters

Recall : For a standard form linear equation $y'' + P(x)y' + Q(x)y = g(x)$ we obtained y_p from $y_c = c_1y_1 + c_2y_2$ by "varying" c_1, c_2 and looking for

$$y_p = u_1(t)y_1(t) + u_2(t)y_2(t).$$

Same idea for a linear system :

- Consider the nonhomogeneous linear system : $\boxed{\vec{x}' = A\vec{x} + \vec{g}(t)}$
- Complementary Solution : Recall that the solution to $\vec{x}' = A\vec{x}$ can be expressed as : $\boxed{\vec{x}_c = \Phi(t)\vec{c}}$ where $\Phi(t)$ is a fundamental matrix and

$$\vec{c} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$ is a vector of arbitrary constants.

- Particular Solution : Replace $\vec{c}$ by $\vec{u}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_n(t) \end{pmatrix}$ so that $\boxed{\vec{x}_p = \Phi(t)\vec{u}(t)}$ is a particular solution to $\vec{x}' = A\vec{x} + \vec{g}(t)$.

$$\vec{x}_p = \Phi(t)\vec{u}(t) \Rightarrow \vec{x}_p' = \Phi'(t)\vec{u}(t) + \Phi(t)\vec{u}'(t)$$

$$\vec{x}_p' = A\vec{x}_p + \vec{g}(t) \Rightarrow \Phi'(t)\vec{u}(t) + \Phi(t)\vec{u}'(t) = A\Phi(t)\vec{u}(t) + \vec{g}(t)$$

$$\boxed{\Phi(t)' = A\Phi(t)} \Rightarrow A\cancel{\Phi(t)\vec{u}(t)} + \Phi(t)\vec{u}'(t) = A\cancel{\Phi(t)\vec{u}(t)} + \vec{g}(t)$$

Why? $\vec{x}_c = \Phi(t)\vec{c}$ is a solution to $\vec{x}' = A\vec{x}$, so

$$\Phi'(t)\vec{c} = A\Phi(t)\vec{c}$$

$$[\Phi'(t) - A\Phi(t)]\vec{c} = \vec{0}$$

Since this is true for all $\vec{c} \in \mathbb{R}^n$, we must have

$$\Phi'(t) = A\Phi(t)$$

$$\Rightarrow \Phi(t)\vec{u}'(t) = \vec{g}(t)$$

$$\Rightarrow \boxed{\vec{u}'(t) = \Phi^{-1}(t)\vec{g}(t)}$$

$$\Rightarrow \boxed{\vec{u}(t) = \int \Phi^{-1}(t)\vec{g}(t)\,dt}$$

$$\Rightarrow \boxed{\vec{x}_p = \Phi(t)\int \Phi^{-1}(t)\vec{g}(t)\,dt}$$

- General Solution : $\boxed{\vec{x} = \vec{x}_c + \vec{x}_p = \Phi(t)\vec{c} + \Phi(t)\int \Phi^{-1}(t)\vec{g}(t)\,dt}$

Example: $\vec{x}' = \begin{pmatrix} -3 & 1 \\ 2 & -4 \end{pmatrix}\vec{x} + \begin{pmatrix} 3t \\ e^{-t} \end{pmatrix}$

- Complementary Sol. $\vec{x}' = A\vec{x}$

$$\begin{vmatrix} -3-\lambda & 1 \\ 2 & -4-\lambda \end{vmatrix} = (\lambda+3)(\lambda+4)-2 = \lambda^2+7\lambda+10 = (\lambda+2)(\lambda+5)$$

Eigenvalues: $\lambda_1 = -2$; $\lambda_2 = -5$

$$[A+2I|\vec{0}] = \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 2 & -2 & 0 \end{array}\right] \Rightarrow v_1 = v_2 \Rightarrow \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \vec{x}_1 = e^{-2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$[A+5I|\vec{0}] = \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 2 & 1 & 0 \end{array}\right] \Rightarrow v_2 = -2v_1 \Rightarrow \vec{v}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \Rightarrow \vec{x}_2 = e^{-5t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$\Rightarrow \boxed{\Phi(t) = \begin{pmatrix} e^{-2t} & e^{-5t} \\ e^{-2t} & -2e^{-5t} \end{pmatrix}} \Rightarrow \det(\Phi(t)) = -3e^{-7t}$$

- Particular Solution:

$$\Phi^{-1}(t) = \frac{1}{-3e^{-7t}}\begin{pmatrix} -2e^{-5t} & -e^{-5t} \\ -e^{-2t} & e^{-2t} \end{pmatrix} = \begin{pmatrix} 2/3\, e^{2t} & 1/3\, e^{2t} \\ 1/3\, e^{5t} & -1/3\, e^{5t} \end{pmatrix}$$

$$\Phi^{-1}(t)\vec{g}(t) = \begin{pmatrix} 2/3\, e^{2t} & 1/3\, e^{2t} \\ 1/3\, e^{5t} & -1/3\, e^{5t} \end{pmatrix}\begin{pmatrix} 3t \\ e^{-t} \end{pmatrix} = \begin{pmatrix} 2te^{2t} + 1/3\, e^{t} \\ te^{5t} - 1/3\, e^{4t} \end{pmatrix}$$

$$\Rightarrow \vec{u}(t) = \int \begin{pmatrix} 2te^{2t} + 1/3\, e^{t} \\ te^{5t} - 1/3\, e^{4t} \end{pmatrix} dt = \begin{pmatrix} te^{2t} - 1/2\, e^{2t} + 1/3\, e^{t} \\ 1/5\, te^{5t} - 1/25\, e^{5t} - 1/12\, e^{4t} \end{pmatrix}$$

$$\Rightarrow \vec{x}_p = \Phi(t)\vec{u}(t) = \begin{pmatrix} e^{-2t} & e^{-5t} \\ e^{-2t} & -2e^{-5t} \end{pmatrix}\begin{pmatrix} te^{2t} - 1/2\, e^{2t} + 1/3\, e^{t} \\ 1/5\, te^{5t} - 1/25\, e^{5t} - 1/12\, e^{4t} \end{pmatrix}$$

$$= \begin{pmatrix} t - 1/2 + 1/3\, e^{-t} + 1/5\, t - 1/25 - 1/12\, e^{-t} \\ t - 1/2 + 1/3\, e^{-t} - 2/5\, t + 2/25 + 1/6\, e^{-t} \end{pmatrix} = \underline{\underline{\begin{pmatrix} 6/5\, t + 1/4\, e^{-t} - 27/50 \\ 3/5\, t + 1/2\, e^{-t} - 21/50 \end{pmatrix}}}$$

Recall: Inverting a matrix using Elementary Row Operations

$$A = \begin{pmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{pmatrix} \quad ; \quad A^{-1} = ?$$

$$\left(\begin{array}{ccc|ccc} 3 & 0 & 2 & 1 & 0 & 0 \\ 2 & 0 & -2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow[\frac{1}{2}R_2]{\frac{1}{3}R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 2/3 & 1/3 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1/2 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[R_2-R_1]{} \left(\begin{array}{ccc|ccc} 1 & 0 & 2/3 & 1/3 & 0 & 0 \\ 0 & 0 & -5/3 & -1/3 & 1/2 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow[-\frac{3}{5}R_2]{} \left(\begin{array}{ccc|ccc} 1 & 0 & 2/3 & 1/3 & 0 & 0 \\ 0 & 0 & 1 & 1/5 & -3/10 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[R_3-R_2]{R_1-\frac{2}{3}R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/5 & 1/5 & 0 \\ 0 & 0 & 1 & 1/5 & -3/10 & 0 \\ 0 & 1 & 0 & -1/5 & 3/10 & 1 \end{array}\right) \xrightarrow[R_2 \leftrightarrow R_3]{} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/5 & 1/5 & 0 \\ 0 & 1 & 0 & -1/5 & 3/10 & 1 \\ 0 & 0 & 1 & 1/5 & -3/10 & 0 \end{array}\right)$$

$$\Rightarrow \boxed{A^{-1} = \begin{pmatrix} 1/5 & 1/5 & 0 \\ -1/5 & 3/10 & 1 \\ 1/5 & -3/10 & 0 \end{pmatrix}}$$

Matrix Exponential

- Recall: if $x(t)$ is a function of t, the equation $x' = ax$ has general solution $x = ce^{at}$.
- Question: Can we somehow extend this to homogeneous linear systems?

That is, can we have the general solution of $\vec{x}' = A\vec{x}$ be of the form $\vec{x} = e^{tA}\vec{c}$?

Definition: Let A be an $n \times n$ matrix. Define the matrix exponential:

$$\boxed{e^{tA} := \sum_{n=0}^{\infty} \frac{(tA)^n}{n!} = I + tA + \frac{t^2}{2}A^2 + \frac{t^3}{3!}A^3 + \ldots} \qquad (*)$$

- The series in $(*)$ converges to an $n \times n$ matrix for all real t.
- The matrix e^{tA} is invertible, with inverse e^{-tA}.
- Take $t=0$ in $(*)$: $\boxed{e^{0 \cdot A} = I}$
- Differentiate both sides of $(*)$: $\boxed{\frac{d}{dt} e^{tA} = A e^{tA}}$

$$\frac{d}{dt} e^{tA} = A + tA^2 + \frac{t^2}{2!}A^3 + \ldots = A\left(I + tA + \frac{t^2}{2!}A^2 + \ldots\right) = Ae^{tA}.$$

$\Rightarrow$ e^{tA} is the special fundamental matrix $\Psi(t)$ of the system $\vec{x}' = A\vec{x}$ that satisfies $\Psi(0) = I$. $\boxed{e^{tA} = \Psi(t)}$

Recall: If $\Phi(t)$ is a fundamental matrix of $\vec{x}' = A\vec{x}$, then $\Phi(t)' = A\Phi(t)$. Conversely, if $\Phi(t)$ is an invertible matrix that satisfies $\Phi'(t) = A\Phi(t)$ for all t in some interval I, then $\Phi(t)$ is a fundamental matrix of $\vec{x}' = A\vec{x}$.

Let $\Psi(t) = e^{tA}$. Then $\boxed{\frac{d}{dt} e^{tA} = Ae^{tA}}$ translates to $\Psi'(t) = A\Psi(t)$. Thus e^{tA} is a fundamental matrix for the system $\vec{x}' = A\vec{x}$. Moreover, we know that $e^{0 \cdot A} = I$, or $\Psi(0) = I$. So $\Psi(t)$ is the (special) fundamental matrix that satisfies $\Psi(0) = I$.

Relationship to Variation of Parameters:

Recall that we obtained the general solution of $\vec{x}' = A\vec{x} + \vec{g}(t)$ to be

$$\vec{x}(t) = \Phi(t)\vec{c} + \Phi(t)\int \Phi^{-1}(t)\vec{g}(t)\,dt$$

where $\Phi(t)$ is a fundamental matrix of the homogeneous system $\vec{x}' = A\vec{x}$. So take $\Phi(t)$ to be $\Psi(t) = e^{tA}$ above:

$$\vec{x}(t) = \Psi(t)\vec{c} + \Psi(t)\int \Psi^{-1}(t)\vec{g}(t)\,dt$$

$$= e^{tA}\vec{c} + e^{tA}\int \underbrace{(e^{tA})^{-1}}_{\text{this is simply } e^{-tA}!}\vec{g}(t)\,dt$$

$$\Rightarrow \boxed{\vec{x}(t) = e^{tA}\vec{c} + e^{tA}\int e^{-tA}\vec{g}(t)\,dt}$$ is the general solution to $\vec{x}' = A\vec{x} + \vec{g}(t)$

$\Rightarrow$ If you know e^{tA}, then you can find the solution to any non-homogeneous linear system without having to invert any matrices! Because $(e^{tA})^{-1} = e^{-tA}$, so all you have to do is replace t by $(-t)$ in e^{-tA} and you have your inverse.

Example: Find e^{tA} for $A = \begin{pmatrix} -3 & 1 \\ 2 & -4 \end{pmatrix}$.

Solve $\vec{x}' = A\vec{x}$: we already did this earlier and found the fundamental matrix

$$\Phi(t) = \begin{pmatrix} e^{-2t} & e^{-5t} \\ e^{-2t} & -2e^{-5t} \end{pmatrix}.$$

Goal : Find $e^{tA} = \Psi(t)$, the fundamental matrix with $\Psi(0) = I$.

Two ways to do this:

I.) Use: $\boxed{\Psi(t) = \Phi(t)\Phi^{-1}(0)}$ (this does involve finding $\Phi^{-1}(t)$).

We already computed $\Phi^{-1}(t) = \frac{1}{3}\begin{pmatrix} 2e^{2t} & e^{2t} \\ e^{5t} & -e^{5t} \end{pmatrix} \Rightarrow \Phi^{-1}(0) = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}$

$$\Rightarrow \Psi(t) = \begin{pmatrix} e^{-2t} & e^{-5t} \\ e^{-2t} & -2e^{-5t} \end{pmatrix} \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}$$

$$e^{tA} = \Psi(t) = \frac{1}{3}\begin{pmatrix} 2e^{-2t} + e^{-5t} & e^{-2t} - e^{-5t} \\ 2e^{-2t} - 2e^{-5t} & e^{-2t} + 2e^{-5t} \end{pmatrix}$$

II.) Solve $\boxed{\vec{x} = \Phi(t)\vec{c}}$ subject to $\boxed{\vec{x}(0) = \vec{e}_1 \,;\, \vec{x}(0) = \vec{e}_2 ; \ldots \vec{x}(0) = \vec{e}_n}$ and obtain the n columns of $\Psi(t)$ (this does not involve inverting matrices)

$$\vec{x}(t) = \Phi(t)\vec{c} \Rightarrow \vec{x}(0) = \Phi(0)\vec{c} = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} c_1 + c_2 \\ c_1 - 2c_2 \end{pmatrix}$$

$\circledast$ $\begin{pmatrix} c_1 + c_2 \\ c_1 - 2c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} c_1 + c_2 = 1 \\ c_1 - 2c_2 = 0 \end{cases}$ $\begin{matrix} 3c_2 = 1 \Rightarrow c_2 = 1/3 \\ c_1 = 2c_2 \Rightarrow c_1 = 2/3 \end{matrix}$ $\Rightarrow \vec{x}_1 = \Phi(t)\begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix}$

$\Rightarrow \vec{x}_1 = \frac{1}{3}\begin{pmatrix} e^{-2t} & e^{-5t} \\ e^{-2t} & -2e^{-5t} \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} \Rightarrow \vec{x}_1 = \frac{1}{3}\begin{pmatrix} 2e^{-2t} + e^{-5t} \\ 2e^{-2t} - 2e^{-5t} \end{pmatrix} \leftarrow$ First column of $\Psi(t)$ ✓

$\circledast$ $\begin{pmatrix} c_1 + c_2 \\ c_1 - 2c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow \begin{cases} c_1 + c_2 = 0 \\ c_1 - 2c_2 = 1 \end{cases}$ $\begin{cases} c_1 = -c_2 \Rightarrow c_1 = 1/3 \\ -3c_2 = 1 \Rightarrow c_2 = -1/3 \end{cases}$ $\Rightarrow \vec{x}_2 = \Phi(t)\begin{pmatrix} 1/3 \\ -1/3 \end{pmatrix}$

$\Rightarrow \vec{x}_2 = \frac{1}{3}\begin{pmatrix} e^{-2t} & e^{-5t} \\ e^{-2t} & -2e^{-5t} \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow \vec{x}_2 = \frac{1}{3}\begin{pmatrix} e^{-2t} - e^{-5t} \\ e^{-2t} + 2e^{-5t} \end{pmatrix} \leftarrow$ Second column of $\Psi(t)$ ✓

Find $e^{-tA} = (e^{tA})^{-1}$

$$e^{tA} = \frac{1}{3}\begin{pmatrix} 2e^{-2t}+e^{-5t} & e^{-2t}-e^{-5t} \\ 2e^{-2t}-2e^{-5t} & e^{-2t}+2e^{-5t} \end{pmatrix} \Rightarrow e^{-tA} = \frac{1}{3}\begin{pmatrix} 2e^{2t}+e^{5t} & e^{2t}-e^{5t} \\ 2e^{2t}-2e^{5t} & e^{2t}+2e^{5t} \end{pmatrix}$$

Use the matrix exponential to solve:

$$\vec{x}' = \begin{pmatrix} -3 & 1 \\ 2 & -4 \end{pmatrix}\vec{x} + \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

General Solution:

$$\boxed{\vec{x}(t) = e^{tA}\vec{c} + e^{tA}\int e^{-tA}\vec{g}(t)\,dt} \qquad \vec{g}(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$e^{-tA}\vec{g}(t) = \frac{1}{3}\begin{pmatrix} e^{2t}+2e^{5t} \\ e^{2t}-4e^{5t} \end{pmatrix} \Rightarrow \int e^{-tA}\vec{g}(t)\,dt = \frac{1}{3}\begin{pmatrix} \frac{1}{2}e^{2t}+\frac{2}{5}e^{5t} \\ \frac{1}{2}e^{2t}-\frac{4}{5}e^{5t} \end{pmatrix}$$

$$\Rightarrow \vec{x}_p = e^{tA}\int e^{-tA}\vec{g}(t)\,dt = \frac{1}{3}e^{-5t}\begin{pmatrix} 2e^{3t}+1 & e^{3t}-1 \\ 2e^{3t}-2 & e^{3t}+2 \end{pmatrix}\left[\frac{1}{6}e^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{2}{15}e^{5t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}\right]$$

$$= \frac{1}{18}e^{-3t}\begin{pmatrix} 3e^{3t} \\ 3e^{3t} \end{pmatrix} + \frac{2}{45}\begin{pmatrix} 3 \\ -6 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{2}{15}\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 3/10 \\ -1/10 \end{pmatrix}$$

$$\Rightarrow \boxed{\vec{x}_p = \frac{1}{10}\begin{pmatrix} 3 \\ -1 \end{pmatrix}}$$

Check: $\vec{x}_p{}' = \vec{0}$

$$A\vec{x}_p + \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -3 & 1 \\ 2 & -4 \end{pmatrix}\frac{1}{10}\begin{pmatrix} 3 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{10}\begin{pmatrix} -10 \\ 10 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \vec{0}$$

$$\Rightarrow A\vec{x}_p + \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \vec{x}_p{}' \; //$$

$$\Rightarrow \boxed{\vec{x}(t) = e^{tA}\vec{c} + \frac{1}{10}\begin{pmatrix} 3 \\ -1 \end{pmatrix}}$$