

Linear Systems of First-Order with Constant Coefficients

ii: Distinct Eigenvalues (with Complex Eigenvalues)

① $\vec{x}' = \begin{pmatrix} 6 & -1 \\ 5 & 4 \end{pmatrix} \vec{x}$

$$\det(A - \lambda I) = \begin{vmatrix} 6-\lambda & -1 \\ 5 & 4-\lambda \end{vmatrix} = \lambda^2 - 10\lambda + 24 + 5 = \lambda^2 - 10\lambda + 29$$

$$\Delta = 100 - 116 = -16$$

$$\lambda = \frac{10 \pm 4i}{2}$$

$$\lambda = 5 \pm 2i$$

$\lambda = 5 + 2i$

$$[A - (5 + 2i)I | \vec{0}] = \left[\begin{array}{cc|c} 1-2i & -1 & 0 \\ 5 & -1-2i & 0 \end{array} \right] \Rightarrow v_2 = (1-2i)v_1$$

$$\Rightarrow \vec{v}_1 = \begin{bmatrix} 1 \\ 1-2i \end{bmatrix}$$

$$\Rightarrow \vec{x} = c_1 e^{5t} \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} \cos(2t) - \begin{bmatrix} 0 \\ -2 \end{bmatrix} \sin(2t) \right) + c_2 e^{5t} \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} \sin(2t) + \begin{bmatrix} 0 \\ -2 \end{bmatrix} \cos(2t) \right)$$

$$= c_1 e^{5t} \begin{bmatrix} \cos(2t) \\ \cos(2t) + 2\sin(2t) \end{bmatrix} + c_2 e^{5t} \begin{bmatrix} \sin(2t) \\ \sin(2t) - 2\cos(2t) \end{bmatrix}$$

② $\vec{x}' = \begin{pmatrix} 6 & -1 \\ 5 & 2 \end{pmatrix} \vec{x}$

$$\det(A - \lambda I) = \begin{vmatrix} 6-\lambda & -1 \\ 5 & 2-\lambda \end{vmatrix} = \lambda^2 - 8\lambda + 12 + 5 = \lambda^2 - 8\lambda + 17$$

$$\Delta = 64 - 68 = -4 \quad \lambda = \frac{8 \pm 2i}{2}$$

$\lambda = 4 + i$

$$\lambda = 4 + i: \left[\begin{array}{cc|c} 2-i & -1 & 0 \\ 5 & -2-i & 0 \end{array} \right] \Rightarrow v_2 = (2-i)v_1$$

$$\Rightarrow \vec{v}_1 = \begin{bmatrix} 1 \\ 2-i \end{bmatrix}$$

$$\vec{x} = c_1 e^{4t} \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} \cos t - \begin{bmatrix} 0 \\ -1 \end{bmatrix} \sin t \right) + c_2 e^{4t} \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} \sin t + \begin{bmatrix} 0 \\ -1 \end{bmatrix} \cos t \right)$$

$$= c_1 e^{4t} \begin{bmatrix} \cos t \\ 2\cos t + \sin t \end{bmatrix} + c_2 e^{4t} \begin{bmatrix} \sin t \\ 2\sin t - \cos t \end{bmatrix}$$

$$\textcircled{3} \quad \vec{x}' = \begin{pmatrix} 5 & 1 \\ -2 & 3 \end{pmatrix} \vec{x}$$

$$\det(A - \lambda I) = \begin{vmatrix} 5-\lambda & 1 \\ -2 & 3-\lambda \end{vmatrix} = \lambda^2 - 8\lambda + 15 + 2 = \lambda^2 - 8\lambda + 17$$

$$\lambda = 4 \pm i$$

$$\lambda = 4 + i: \quad \left[\begin{array}{cc|c} 1-i & 1 & 0 \\ -2 & -1-i & 0 \end{array} \right] \Rightarrow v_2 = (-1+i)v_1 \Rightarrow \vec{v}_1 = \begin{bmatrix} 1 \\ -1+i \end{bmatrix}$$

$$\vec{x} = c_1 e^{4t} \left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} \cos t - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sin t \right) + c_2 e^{4t} \left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} \sin t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cos t \right)$$

$$= c_1 e^{4t} \begin{bmatrix} \cos t \\ -\cos t - \sin t \end{bmatrix} + c_2 e^{4t} \begin{bmatrix} \sin t \\ -\sin t + \cos t \end{bmatrix}$$

$$\textcircled{4} \quad \vec{x}' = \begin{pmatrix} 4 & -5 \\ 5 & -4 \end{pmatrix} \vec{x}$$

$$\det(A - \lambda I) = \begin{vmatrix} 4-\lambda & -5 \\ 5 & -4-\lambda \end{vmatrix} = \lambda^2 - 16 + 25 = \lambda^2 + 9$$

$$\lambda = \pm 3i$$

$$\lambda = 3i: \quad \left[\begin{array}{cc|c} 4-3i & -5 & 0 \\ 5 & -4-3i & 0 \end{array} \right] \Rightarrow v_2 = \frac{4-3i}{5} v_1$$

$$\vec{v}_1 = \begin{bmatrix} 5 \\ 4-3i \end{bmatrix}$$

$$\vec{x} = c_1 e^0 \left(\begin{bmatrix} 5 \\ 4 \end{bmatrix} \cos(3t) - \begin{bmatrix} 0 \\ -3 \end{bmatrix} \sin(3t) \right) + c_2 e^0 \left(\begin{bmatrix} 5 \\ 4 \end{bmatrix} \sin(3t) + \begin{bmatrix} 0 \\ -3 \end{bmatrix} \cos(3t) \right)$$

$$= c_1 \begin{bmatrix} 5 \cos(3t) \\ 4 \cos(3t) + 3 \sin(3t) \end{bmatrix} + c_2 \begin{bmatrix} 5 \sin(3t) \\ 4 \sin(3t) - 3 \cos(3t) \end{bmatrix}$$

$$\textcircled{5} \begin{cases} x'(t) = z \\ y'(t) = -z \\ z'(t) = y \end{cases} \quad \vec{x}' = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \vec{x}$$

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 0 & 1 \\ 0 & -\lambda & -1 \\ 0 & 1 & -\lambda \end{vmatrix} = -\lambda(\lambda^2 + 1) \Rightarrow \lambda \in \{0, \pm i\}$$

$$\lambda = 0: \begin{bmatrix} 0 & 0 & 1 & | & 0 \\ 0 & 0 & -1 & | & 0 \\ 0 & 1 & 0 & | & 0 \end{bmatrix} \quad \begin{matrix} v_3 = 0 \\ v_2 = 0 \end{matrix} \Rightarrow \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\lambda = i: \begin{bmatrix} -i & 0 & 1 & | & 0 \\ 0 & -i & -1 & | & 0 \\ 0 & 1 & -i & | & 0 \end{bmatrix} \xrightarrow{iR_1, iR_2} \begin{bmatrix} 1 & 0 & i & | & 0 \\ 0 & 1 & -i & | & 0 \\ 0 & 1 & -i & | & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & i & | & 0 \\ 0 & 1 & -i & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{matrix} v_1 = -i v_3 \\ v_2 = i v_3 \end{matrix}$$

$$\vec{v}_2 = \begin{pmatrix} 1 \\ -1 \\ i \end{pmatrix}$$

$$\vec{x} = c_1 e^0 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 e^0 \left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \cos t - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \sin t \right) + c_3 e^0 \left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \sin t + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cos t \right)$$

$$\vec{x} = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} \cos t \\ -\cos t \\ -\sin t \end{bmatrix} + c_3 \begin{bmatrix} \sin t \\ -\sin t \\ \cos t \end{bmatrix}$$

$$\textcircled{6} \quad \vec{x}' = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \vec{x}$$

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -1 & 2 \\ -1 & 1-\lambda & 0 \\ -1 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)((\lambda-1)^2 - 1) + 2(1-\lambda) \\ = (1-\lambda)(\lambda^2 - 2\lambda + 2)$$

$$\lambda \in \{1, 1 \pm i\}$$

$$\lambda = 1:$$

$$\Delta = 4 - 8 = -4 \Rightarrow \frac{2 \pm 2i}{2}$$

$$\left[\begin{array}{ccc|c} 0 & -1 & 2 & 0 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{array} \right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$v_1 = 0 \\ v_2 = 2v_3$$

$$\vec{v}_1 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

$$\lambda = 1+i:$$

$$\left[\begin{array}{ccc|c} -i & -1 & 2 & 0 \\ -1 & -i & 0 & 0 \\ -1 & 0 & -i & 0 \end{array} \right] \xrightarrow{\substack{iR_1 \\ -R_2 \\ -R_3}} \left[\begin{array}{ccc|c} 1 & -i & 2i & 0 \\ 1 & i & 0 & 0 \\ 1 & 0 & i & 0 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & -i & 2i & 0 \\ 0 & +2i & -2i & 0 \\ 0 & i & -i & 0 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & -i & 2i & 0 \\ 0 & i & -i & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & i & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \quad \begin{matrix} v_1 = -i v_3 \\ v_2 = v_3 \end{matrix}$$

$$\vec{v}_2 = \begin{bmatrix} 1 \\ i \\ i \end{bmatrix}$$

$$\vec{x} = c_1 e^t \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} + c_2 e^t \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cos t - \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \sin t \right) + c_3 e^t \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \sin t + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \cos t \right)$$

$$\vec{x} = c_1 e^t \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} + c_2 e^t \begin{bmatrix} \cos t \\ -\sin t \\ -\sin t \end{bmatrix} + c_3 e^t \begin{bmatrix} \sin t \\ \cos t \\ \cos t \end{bmatrix}$$



$$\vec{x}' = \begin{pmatrix} 2 & 5 & 1 \\ -5 & -6 & 4 \\ 0 & 0 & 2 \end{pmatrix} \vec{x}$$

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 5 & 1 \\ -5 & -6-\lambda & 4 \\ 0 & 0 & 2-\lambda \end{vmatrix} = (2-\lambda) \begin{vmatrix} 2-\lambda & 5 \\ -5 & -6-\lambda \end{vmatrix} = (2-\lambda) (\lambda^2 + 4\lambda + 13)$$

$$\Delta = 16 - 52 = -36$$

$$\lambda \in \left\{ 2, \frac{-4 \pm 6i}{2} \right\}$$

$$\lambda \in \{ 2, -2 \pm 3i \}$$

$$\lambda = 2:$$

$$[A - 2I | \vec{0}] = \left[\begin{array}{ccc|c} 0 & 5 & 1 & 0 \\ -5 & -8 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & 8/5 & -4/5 & 0 \\ 0 & 1 & 1/5 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -28/25 & 0 \\ 0 & 1 & 1/5 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$v_1 = \frac{28}{25} v_3; \quad v_2 = -\frac{1}{5} v_3$$

$$\vec{v}_1 = \begin{bmatrix} 28 \\ -5 \\ 25 \end{bmatrix}$$

$$\lambda = -2 - 3i$$

$$[A + (2+3i)I | \vec{0}] = \left[\begin{array}{ccc|c} 4+3i & 5 & 1 & 0 \\ -5 & -4+3i & 4 & 0 \\ 0 & 0 & 4+3i & 0 \end{array} \right] \Rightarrow v_1 = -\frac{5}{4+3i} v_2$$

$$\vec{v}_2 = \begin{bmatrix} 5 \\ -4-3i \\ 0 \end{bmatrix}$$

$$\vec{x} = c_1 \begin{bmatrix} 28 \\ -5 \\ 25 \end{bmatrix} e^{2t} + c_2 e^{-2t} \begin{bmatrix} 5 \cos(3t) \\ -4 \cos(3t) + 3 \sin(3t) \\ 0 \end{bmatrix} + c_3 e^{-2t} \begin{bmatrix} 5 \sin(3t) \\ -4 \sin(3t) - 3 \cos(3t) \\ 0 \end{bmatrix}$$

Repeated Eigenvalues

$$\textcircled{8} \quad \vec{x}' = \begin{pmatrix} 4 & -1 \\ 1 & 2 \end{pmatrix} \vec{x}$$

$$\begin{vmatrix} 4-\lambda & -1 \\ 1 & 2-\lambda \end{vmatrix} = (\lambda-4)(\lambda-2)+1 = \lambda^2 - 6\lambda + 9 = (\lambda-3)^2$$

$\lambda=3$ \rightarrow algebraic multiplicity 2

$$[A-3I | \vec{0}] = \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & -1 & 0 \end{array} \right] \Rightarrow v_1 = v_2$$

$$\vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \vec{x}_1 = e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

\hookrightarrow geometric multiplicity 1

Second solution?

$$(A-3I)\vec{w} = \vec{v} \Rightarrow \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow w_1 - w_2 = 1 \Rightarrow w_1 = w_2 + 1$$

$$\vec{w} = \begin{pmatrix} w_2 + 1 \\ w_2 \end{pmatrix} \quad \vec{w} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\vec{x}_2 = t e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{3t} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\Rightarrow \vec{x} = c_1 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \left[t e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{3t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right]$$

or
 \Rightarrow

$$\vec{x} = e^{3t} \begin{pmatrix} c_1 + c_2 t + c_2 \\ c_1 + c_2 t \end{pmatrix}$$

$$(9) \quad \vec{x}' = \begin{pmatrix} 7 & 1 \\ -4 & 3 \end{pmatrix} \vec{x}$$

$$\begin{vmatrix} 7-\lambda & 1 \\ -4 & 3-\lambda \end{vmatrix} = \lambda^2 - 10\lambda + 21 + 4 = (\lambda - 5)^2$$

$$[A - 5I | \vec{0}] = \left[\begin{array}{cc|c} 2 & 1 & 0 \\ -4 & -2 & 0 \end{array} \right] \quad \begin{array}{l} 2v_1 + v_2 = 0 \\ v_2 = -2v_1 \end{array}$$

$$\vec{v} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$\vec{x}_1 = e^{5t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Second Solution:

$$(A - 5I) \vec{w} = \vec{v} \quad \begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad 2w_1 + w_2 = 1$$

Take $\vec{w} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\vec{x}_2 = e^{5t} \cdot t \begin{pmatrix} 1 \\ -2 \end{pmatrix} + e^{5t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\vec{x} = c_1 e^{5t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2 \left[t e^{5t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + e^{5t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]$$

$$\text{or } \vec{x} = e^{5t} \begin{pmatrix} c_1 + t c_2 \\ -2c_1 - 2t c_2 + c_2 \end{pmatrix}$$

$$(10) \quad \vec{x}' = \begin{pmatrix} -1 & 3/2 \\ -1/6 & -2 \end{pmatrix} \vec{x}$$

$$\begin{vmatrix} -1-\lambda & 3/2 \\ -1/6 & -2-\lambda \end{vmatrix} = \lambda^2 + 3\lambda + 1 + \frac{1}{4} = \lambda^2 + 3\lambda + \frac{9}{4} = (\lambda + 3/2)^2 \quad \lambda = -3/2$$

$$[A + 3/2 I | \vec{0}] = \left[\begin{array}{cc|c} 1/2 & 3/2 & 0 \\ -1/6 & -1/2 & 0 \end{array} \right]$$

$$v_1 + 3v_2 = 0 \Rightarrow \vec{v} = \begin{pmatrix} -3 \\ 1 \end{pmatrix} \Rightarrow \vec{x}_1 = e^{-3/2 t} \begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

$$(A + 3/2 I) \vec{w} = \vec{v}$$

$$\begin{pmatrix} 1/2 & 3/2 \\ -1/6 & -1/2 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \end{pmatrix} \Rightarrow \frac{1}{2} w_1 + \frac{3}{2} w_2 = -3$$

$$\begin{array}{l} (w_2 = 2): \\ \vec{w} = \begin{pmatrix} -12 \\ 2 \end{pmatrix} \end{array} \quad \text{or} \quad \begin{array}{l} (w_2 = 0): \\ \vec{w} = \begin{pmatrix} -6 \\ 0 \end{pmatrix} \end{array} \quad \text{etc.}$$

$$\Rightarrow \vec{x}_2 = e^{-3/2 t} \cdot t \begin{pmatrix} -3 \\ 1 \end{pmatrix} + e^{-3/2 t} \begin{pmatrix} -12 \\ 2 \end{pmatrix}$$

$$\Rightarrow \vec{x} = e^{-3/2 t} \begin{pmatrix} -3c_1 - 3c_2 t - 12c_2 \\ c_1 + c_2 t + 2c_2 \end{pmatrix}$$

$$\textcircled{11} \quad \vec{x}' = \begin{pmatrix} 2 & 1/2 \\ -1/2 & 1 \end{pmatrix} \vec{x}$$

$$\begin{vmatrix} 2-\lambda & 1/2 \\ -1/2 & 1-\lambda \end{vmatrix} = \lambda^2 - 3\lambda + 2 + \frac{1}{4} = (\lambda - 3/2)^2$$

$$[A - 3/2 I | \vec{0}] = \left[\begin{array}{cc|c} 1/2 & 1/2 & 0 \\ -1/2 & -1/2 & 0 \end{array} \right] \quad v_1 = -v_2 \quad \boxed{\vec{v} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}}$$

$$(A - \lambda I) \vec{w} = \vec{v} \quad \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & -1/2 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \begin{array}{l} \frac{1}{2}(w_1 + w_2) = -1 \\ w_1 + w_2 = -2 \end{array} \quad \boxed{\vec{w} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}}$$

$$\begin{aligned} \vec{x} &= c_1 e^{3/2 t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 \left(t e^{3/2 t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + e^{3/2 t} \begin{pmatrix} -1 \\ -1 \end{pmatrix} \right) \\ &= e^{3/2 t} \begin{pmatrix} -c_1 - c_2 - c_2 t \\ c_1 - c_2 + c_2 t \end{pmatrix} \end{aligned}$$

$$\textcircled{12} \quad A = \begin{pmatrix} 1 & -2 & 2 \\ -2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$$

$$\begin{vmatrix} 1-\lambda & -2 & 2 \\ -2 & 1-\lambda & -2 \\ 2 & -2 & 1-\lambda \end{vmatrix} \stackrel{R_1+R_2}{=} \begin{vmatrix} -1-\lambda & -1-\lambda & 0 \\ -2 & 1-\lambda & -2 \\ 2 & -2 & 1-\lambda \end{vmatrix} = \begin{vmatrix} \lambda+1 & \lambda+1 & 0 \\ 2 & \lambda-1 & 2 \\ 2 & -2 & 1-\lambda \end{vmatrix} \stackrel{c_2 \leftarrow c_1}{=} \begin{vmatrix} \lambda+1 & 0 & 0 \\ 2 & \lambda-3 & 2 \\ 2 & -4 & 1-\lambda \end{vmatrix}$$

$$= (\lambda+1)(-\lambda^2 + 4\lambda - 3 + 8) = (\lambda+1)(-\lambda^2 + 4\lambda + 5)$$

$$= -(\lambda+1)(\lambda-5)(\lambda+1)$$

$$= -(\lambda+1)^2(\lambda-5)$$

Eigenvalues: $\lambda_1 = -1$; $\lambda_2 = -1$; $\lambda_3 = 5$.

$$\Rightarrow \boxed{\text{alg}(\lambda=5) = \text{geom}(\lambda=5) = 1}$$

$$\Rightarrow \boxed{\text{alg}(\lambda=-1) = 2}$$

$$[A+I|\vec{0}] = \left[\begin{array}{ccc|c} 2 & -2 & 2 & 0 \\ -2 & 2 & -2 & 0 \\ 2 & -2 & 2 & 0 \end{array} \right] \xrightarrow{\substack{R_1+R_2 \\ R_3-R_1}} \left[\begin{array}{ccc|c} 2 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \xrightarrow{\frac{1}{2}R_1} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$v_1 - v_2 + v_3 = 0$$

(3 unknowns, 1 restriction) \Rightarrow we can find 2 linearly indep. eigenvectors

For example:

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \quad \& \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\Rightarrow \boxed{\text{geom}(\lambda=-1) = 2}$$

\Rightarrow Matrix is non-defective

Eigenvector for $\lambda=5$:

$$[A-5I|\vec{0}] = \left[\begin{array}{ccc|c} -4 & -2 & 2 & 0 \\ -2 & -4 & -2 & 0 \\ +2 & -2 & -4 & 0 \end{array} \right] \xrightarrow{\substack{-\frac{1}{4}R_1 \\ -\frac{1}{2}R_2 \\ \frac{1}{2}R_3}} \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 1 & 2 & 1 & 0 \\ 1 & -1 & -2 & 0 \end{array} \right] \xrightarrow{\substack{R_2-R_1 \\ R_3-R_1}} \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & \frac{3}{2} & \frac{3}{2} & 0 \\ 0 & -\frac{3}{2} & -\frac{3}{2} & 0 \end{array} \right]$$

$$\xrightarrow{\substack{R_3+R_2 \\ \frac{2}{3}R_2}} \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \xrightarrow{R_1 - \frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\begin{aligned} v_1 &= v_3 \\ v_2 &= -v_3 \end{aligned}$$

$$\vec{v}_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

Solution to $\vec{x}' = A\vec{x}$:

$$\vec{x} = c_1 e^{-t} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_3 e^{5t} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

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$$A = \begin{pmatrix} 2 & 1 & 6 \\ 0 & 2 & 5 \\ 0 & 0 & 2 \end{pmatrix}$$

Eigenvalues: $\lambda_1 = \lambda_2 = \lambda_3 = 2 \Rightarrow \text{alg}(\lambda=2) = 3$

$$[A - 2I | \vec{0}] = \left[\begin{array}{ccc|c} 0 & 1 & 6 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \Rightarrow v_2 + 6v_3 = 0 \Rightarrow v_2 = 0$$

$$\Rightarrow v_3 = 0 \Rightarrow \vec{v} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow \text{geom}(\lambda=2) = 1 \Rightarrow$ defective matrix

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$$A = \begin{pmatrix} 3 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$$

$$\left[\begin{array}{ccc|c} 3-\lambda & -1 & -1 & 0 \\ 1 & 1-\lambda & -1 & 0 \\ 1 & -1 & 1-\lambda & 0 \end{array} \right] \xrightarrow{\substack{R_3-R_2 \\ R_1-R_2}} \left[\begin{array}{ccc|c} 2-\lambda & \lambda-2 & 0 & 0 \\ 1 & 1-\lambda & -1 & 0 \\ 0 & \lambda-2 & \lambda-2 & 0 \end{array} \right] \xrightarrow{C_2+C_1} \left[\begin{array}{ccc|c} 2-\lambda & 0 & 0 & 0 \\ 1 & 2-\lambda & -1 & 0 \\ 0 & \lambda-2 & \lambda-2 & 0 \end{array} \right]$$

$$= (2-\lambda) (-(\lambda-2)^2 + (\lambda-2)) = (\lambda-2)(\lambda-2)(\lambda-2+1) = (\lambda-2)^2(\lambda-1)$$

$\lambda_1 = \lambda_2 = 2$; $\lambda_3 = 1 \Rightarrow \text{alg}(\lambda=1) = \text{geom}(\lambda=1) = 1$; $\text{alg}(\lambda=2) = 2$

$$[A - 2I | \vec{0}] = \left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 1 & -1 & -1 & 0 \\ 1 & -1 & -1 & 0 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \Rightarrow v_1 - v_2 - v_3 = 0$$

$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}; \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$\text{geom}(\lambda=2) = 2 \Rightarrow$ non-defective

$$[A - I | \vec{0}] = \left[\begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & -1/2 & -1/2 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & -1/2 & -1/2 & 0 \\ 0 & -1/2 & 1/2 & 0 \\ 0 & 1/2 & -1/2 & 0 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & -1/2 & -1/2 & 0 \\ 0 & -1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \quad v_1 = v_3, v_2 = v_3 \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Solution to $\vec{x}' = A\vec{x}$:

$$\vec{x} = c_1 e^{2t} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_3 e^t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$