Math 2552 - Differential Equations **Homework 20**
Sections F1 – F4; L1 – L4
Georgia Institute of Technology, Fall 2015

First-Order Linear Systems of Differential Equations with Constant Coefficients
II. Complex Eigenvalues

Solve the following systems:

1. $\mathbf{x}' = \begin{pmatrix} 6 & -1 \\ 5 & 4 \end{pmatrix} \mathbf{x}.$

2. $\mathbf{x}' = \begin{pmatrix} 6 & -1 \\ 5 & 2 \end{pmatrix} \mathbf{x}.$

3. $\mathbf{x}' = \begin{pmatrix} 5 & 1 \\ -2 & 3 \end{pmatrix} \mathbf{x}.$

4. $\mathbf{x}' = \begin{pmatrix} 4 & -5 \\ 5 & -4 \end{pmatrix} \mathbf{x}.$

5. $\begin{cases} x'(t) = z \\ y'(t) = -z \\ z'(t) = y. \end{cases}$

6. $\mathbf{x}' = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \mathbf{x}.$

7. $\mathbf{x}' = \begin{pmatrix} 2 & 5 & 1 \\ -5 & -6 & 4 \\ 0 & 0 & 2 \end{pmatrix} \mathbf{x}.$

First-Order Linear Systems of Differential Equations with Constant Coefficients
III. Repeated Eigenvalues

8. $\mathbf{x}' = \begin{pmatrix} 4 & -1 \\ 1 & 2 \end{pmatrix} \mathbf{x}.$

9. $\mathbf{x}' = \begin{pmatrix} 7 & 1 \\ -4 & 3 \end{pmatrix} \mathbf{x}.$

10. $\mathbf{x}' = \begin{pmatrix} -1 & 3/2 \\ -1/6 & -2 \end{pmatrix} \mathbf{x}.$

11. $\mathbf{x}' = \begin{pmatrix} 2 & 1/2 \\ -1/2 & 1 \end{pmatrix} \mathbf{x}.$

For each of the matrices below, find the eigenvalues and, for each eigenvalue, find its algebraic and its geometric multiplicity. If the matrix A is non-defective, solve the system $\mathbf{x}' = A\mathbf{x}$.

12. $A = \begin{pmatrix} 1 & -2 & 2 \\ -2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}.$

13. $A = \begin{pmatrix} 2 & 1 & 6 \\ 0 & 2 & 5 \\ 0 & 0 & 2 \end{pmatrix}.$

14. $A = \begin{pmatrix} 3 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}.$

Complex Eigenvalues

Consider a homogeneous linear system $\vec{x}' = A\vec{x}$ with constant coefficients. Suppose that

$$\boxed{\lambda = \alpha + i\beta} \quad \& \quad \boxed{\bar{\lambda} = \alpha - i\beta}$$

is a conjugate pair of eigenvalues of the matrix A. Then their corresponding eigenvectors also occur in conjugate pairs. That is, suppose that

$$\boxed{\vec{v} = \vec{a} + i\vec{b}}$$

is an eigenvector corresponding to λ. Then $\bar{\vec{v}}$ is an eigenvector for $\bar{\lambda}$. Therefore

$$\vec{u}(t) = \boxed{e^{\lambda t}\vec{v}} \quad \text{and} \quad \boxed{e^{\bar{\lambda} t}\bar{\vec{v}}} = \bar{\vec{u}}(t)$$

are solutions to the system. In terms of real-valued functions:

$$\boxed{\begin{aligned} \vec{x}_1(t) &= \mathrm{Re}\,\vec{u}(t) = e^{\alpha t}\left(\vec{a}\cos(\beta t) - \vec{b}\sin(\beta t)\right) \\ \vec{x}_2(t) &= \mathrm{Im}\,\vec{u}(t) = e^{\alpha t}\left(\vec{a}\sin(\beta t) + \vec{b}\cos(\beta t)\right) \end{aligned}}$$

are real-valued, linearly independent solutions.

$$\vec{u}(t) = e^{\lambda t}(\vec{a} + i\vec{b}) = e^{\alpha t}e^{i\beta t}(\vec{a} + i\vec{b}) = e^{\alpha t}\left(\cos(\beta t) + i\sin(\beta t)\right)(\vec{a} + i\vec{b})$$

$$= e^{\alpha t}\left[\left(\vec{a}\cos(\beta t) - \vec{b}\sin(\beta t)\right) + i\left(\vec{a}\sin(\beta t) + \vec{b}\cos(\beta t)\right)\right]$$

$$\Rightarrow \mathrm{Re}\,\vec{u}(t) = e^{\alpha t}\left(\vec{a}\cos(\beta t) - \vec{b}\sin(\beta t)\right) = \frac{1}{2}\left(\vec{u}(t) + \bar{\vec{u}}(t)\right)$$

$$\Rightarrow \mathrm{Im}\,\vec{u}(t) = e^{\alpha t}\left(\vec{a}\sin(\beta t) + \vec{b}\cos(\beta t)\right) = \frac{i}{2}\left(-\vec{u}(t) + \bar{\vec{u}}(t)\right)$$

both solutions to the system

$\Rightarrow$ If a pair $\lambda, \bar{\lambda}$ of complex eigenvalues occurs, the general solution must contain the terms

$$c_1\vec{x}_1(t) \quad \text{and} \quad c_2\vec{x}_2(t)$$

where $\vec{x}_{1,2}(t)$ are as above.

Example: $\vec{x}' = \begin{pmatrix} 2 & 8 \\ -1 & -2 \end{pmatrix} \vec{x}$

$$A = \begin{pmatrix} 2 & 8 \\ -1 & -2 \end{pmatrix} \Rightarrow \det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 8 \\ -1 & -2-\lambda \end{vmatrix} = -(4-\lambda^2) + 8 = \lambda^2 + 4$$

$$\Rightarrow \boxed{\lambda = \pm 2i}$$

Eigenvector for $\lambda = 2i$:

$$[A - 2iI \mid \vec{0}] = \left[\begin{array}{cc|c} 2-2i & 8 & 0 \\ -1 & -2-2i & 0 \end{array}\right]$$

$$\begin{cases} (2-2i)v_1 + 8v_2 = 0 \\ -v_1 - (2+2i)v_2 = 0 \Rightarrow \boxed{v_1 = -(2+2i)v_2} \end{cases} \Rightarrow \boxed{\vec{v} = \begin{bmatrix} 2+2i \\ -1 \end{bmatrix}}$$

$$\Rightarrow (2-2i)(2+2i)v_2 - 8v_2 = 0$$

$$\Rightarrow (4+4)v_2 - 8v_2 = 0 \Rightarrow 8v_2 - 8v_2 = 0 \checkmark$$

$$\Rightarrow \boxed{\operatorname{Re}\vec{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}; \quad \operatorname{Im}\vec{v} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}}$$

$\Rightarrow$ General Solution:

$$\vec{x} = c_1\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}\cos(2t) - \begin{bmatrix} 2 \\ 0 \end{bmatrix}\sin(2t)\right) + c_2\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}\sin(2t) + \begin{bmatrix} 2 \\ 0 \end{bmatrix}\cos(2t)\right)$$

$$= c_1\begin{bmatrix} 2\cos(2t) - 2\sin(2t) \\ -\cos(2t) \end{bmatrix} + c_2\begin{bmatrix} 2\sin(2t) + 2\cos(2t) \\ -\sin(2t) \end{bmatrix}$$

Homogeneous Linear Systems with Constant Coefficients: Repeated Eigenvalues

Let λ be an eigenvalue of a matrix A:

- The **algebraic multiplicity** of λ is the multiplicity of λ as a solution to the characteristic equation of A.
- The **geometric multiplicity** of λ is the dimension of its eigenspace (the number of linearly independent eigenvectors λ can have).

If $\text{alg.}(\lambda) = \text{geom.}(\lambda)$, the eigenvalue λ is said to be **nondefective**.
Otherwise, if $\text{geom.}(\lambda) < \text{alg.}(\lambda)$, we say λ is **defective**.
A matrix which has a defective eigenvalue is called a **defective matrix**.

The 2×2 Case:

Example 1: $\vec{x}' = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \vec{x}$

Characteristic Equation: $\begin{vmatrix} 2-\lambda & 0 \\ 0 & 2-\lambda \end{vmatrix} = 0 \Rightarrow (2-\lambda)^2 = 0 \Rightarrow \boxed{\lambda = 2}$ (algebraic multiplicity = 2.)

Eigenvectors?

$$[A-2I|\vec{0}] = \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow$ No restrictions on the solutions!

$\Rightarrow$ Any $\vec{v} \in \mathbb{R}^2$ is an eigenvector! (geometric multiplicity = 2)

Choose for example $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and we have two linearly independent vectors.

$\Rightarrow$ General Solution: $\vec{x} = c_1 e^{2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad \boxed{\vec{x} = e^{2t} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}}$

($\lambda = 2$ is nondefective).

Example 2: $\vec{x}' = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \vec{x}$

Characteristic Equation: $\begin{vmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{vmatrix} = 0 \Rightarrow (2-\lambda)^2 = 0 \Rightarrow \boxed{\lambda = 2}$ (algebraic multiplicity $= 2$)

Eigenvectors?

$[A - 2I \mid \vec{0}] = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow$ One restriction: $v_2 = 0$

$\Rightarrow$ Eigenvector: $\boxed{\vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}}$ (geometric multiplicity $= 1$)

$\Rightarrow \lambda = 2$ is a defective eigenvalue.

We only have $c_1 e^{2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ to put in the general solution.
How do we find a second solution?

Suppose A is a 2×2 matrix and λ is an eigenvalue of algebraic multiplicity 2 and geometric multiplicity 1.
Solve $\vec{x}' = A\vec{x}$:

1. Find an eigenvector $\vec{v}$ corresponding to λ. $\Rightarrow \vec{x}_1 = e^{\lambda t} \vec{v}$
2. Second Solution: a solution of the form:

$$\boxed{\vec{x}_2 = t e^{\lambda t} \vec{v} + e^{\lambda t} \vec{w}}$$

may be found, where $\vec{v}$ is the eigenvector in 1, and $\vec{w}$ is a vector that satisfies:

$$\boxed{(A - \lambda I)\vec{w} = \vec{v}}$$

General Solution:

$$\vec{x} = c_1 \vec{x}_1 + c_2 \vec{x}_2$$

Proof: Consider a vector $\vec{x}_2 = t e^{\lambda t} \vec{v} + e^{\lambda t} \vec{w}$. Then

$\vec{x}_2' = A\vec{x}_2 \iff e^{\lambda t}\vec{v} + \lambda t e^{\lambda t} \vec{v} + \lambda e^{\lambda t} \vec{w} = t e^{\lambda t} \underbrace{A\vec{v}}_{=\lambda \vec{v}} + e^{\lambda t} A\vec{w}$

$\iff e^{\lambda t}\vec{v} + \cancel{\lambda t e^{\lambda t} \vec{v}} + \lambda e^{\lambda t} \vec{w} = \cancel{\lambda t e^{\lambda t} \vec{v}} + e^{\lambda t} A \vec{w}$

$\iff e^{\lambda t} \vec{v} = e^{\lambda t} A\vec{w} - \lambda e^{\lambda t} \vec{w}$

$\iff \vec{v} = A\vec{w} - \lambda \vec{w}$

$\iff \vec{v} = (A - \lambda I)\vec{w}.$

Example 2 finished: $\vec{x}' = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \vec{x}$ $\boxed{\lambda = 2}$

Found: $\boxed{\vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}}$ $\Rightarrow \vec{x}_1 = e^{2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Second Solution: $\vec{x}_2 = te^{2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + e^{2t} \vec{w}$

Find $\vec{w}$: $(A - 2I)\vec{w} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \vec{w} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} w_2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow w_2 = 1$$

$\Rightarrow \vec{w}$ can be any vector of the form $\begin{pmatrix} c \\ 1 \end{pmatrix}$ with $c \in \mathbb{C}$.

Choose $c = 0$: $\vec{w} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\Rightarrow \vec{x}_2 = te^{2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + e^{2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

$$\Rightarrow \vec{x} = c_1 e^{2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \left[te^{2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + e^{2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]$$

$$\vec{x} = e^{2t} \left[\begin{pmatrix} c_1 \\ 0 \end{pmatrix} + \begin{pmatrix} c_2 t \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ c_2 \end{pmatrix} \right] \Rightarrow \boxed{\vec{x} = e^{2t} \begin{pmatrix} c_1 + c_2 t \\ c_2 \end{pmatrix}}$$