Homework (19)
- Solutions -

Homogeneous Linear Systems w/ Constant Coefficients (I)

- Real & Non-repeated Eigenvalues -

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$$\begin{cases} \dfrac{dx}{dt} = -4x+y+z \\ \dfrac{dy}{dt} = x+5y-z \\ \dfrac{dz}{dt} = y-3z \end{cases}$$

$$A = \begin{bmatrix} -4 & 1 & 1 \\ 1 & 5 & -1 \\ 0 & 1 & -3 \end{bmatrix}$$

$$\det(A-\lambda I) = \begin{vmatrix} -4-\lambda & 1 & 1 \\ 1 & 5-\lambda & -1 \\ 0 & 1 & -3-\lambda \end{vmatrix}$$

$$= -(4+\lambda)\begin{vmatrix} 5-\lambda & -1 \\ 1 & -3-\lambda \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 1 & -3-\lambda \end{vmatrix}$$

$$= -(4+\lambda)(\lambda^2-2\lambda-14)+4+\lambda = -\lambda^3-2\lambda^2+23\lambda+60$$

Char. Eqn.: $\boxed{\lambda^3+2\lambda^2-23\lambda-60=0}$

$$\lambda^3+3\lambda^2-\lambda^2-3\lambda-20\lambda-60=0$$

$$(\lambda+3)(\lambda^2-\lambda-20)=0 \Rightarrow (\lambda+3)(\lambda-5)(\lambda+4)=0 \Rightarrow \text{eigenvalues}: -3, 5, -4$$

$\lambda=-3$:

$$[A+3I|\vec{0}] = \left[\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 1 & 8 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \xrightarrow{-R_1} \left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 1 & 8 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \xrightarrow{R_2-R_1} \left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 9 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{1/9 R_2} \left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \xrightarrow{R_3-R_2} \left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1+R_2} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$V_1 = V_3$
$V_2 = 0$

$$\vec{V}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$\lambda=5$:

$$[A-5I|\vec{0}] = \left[\begin{array}{ccc|c} -9 & 1 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & -8 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -1/9 & -1/9 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & -8 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -1/9 & -1/9 & 0 \\ 0 & 1/9 & -8/9 & 0 \\ 0 & 1 & -8 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & -1/9 & -1/9 & 0 \\ 0 & 1 & -8 & 0 \\ 0 & 1 & -8 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -1/9 & -1/9 & 0 \\ 0 & 1 & -8 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -8 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$V_1 = V_3$
$V_2 = 8V_3$

$$\vec{V}_2 = \begin{bmatrix} 1 \\ 8 \\ 1 \end{bmatrix}$$

$\lambda=-4$:

$$[A+4I|\vec{0}] = \left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 1 & 9 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 9 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -10 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$v_1 = 10 v_3$
$v_2 = -v_3$

$$\vec{v_3} = \begin{bmatrix} 10 \\ -1 \\ 1 \end{bmatrix}$$

=> General Solution :

$$\boxed{\vec{x} = c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} 1 \\ 8 \\ 1 \end{bmatrix} e^{5t} + c_3 \begin{bmatrix} 10 \\ -1 \\ 1 \end{bmatrix} e^{-4t}}$$

(2) $\vec{x}' = \begin{pmatrix} 10 & -5 \\ 8 & -12 \end{pmatrix} \vec{x}$

$$\det(A-\lambda I) = \begin{vmatrix} 10-\lambda & -5 \\ 8 & -12-\lambda \end{vmatrix} = -(10-\lambda)(12+\lambda)+40$$
$$= \lambda^2+2\lambda-80$$
$$= (\lambda-8)(\lambda+10) \Rightarrow \lambda = 8 ; -10$$

$\lambda=8$:

$$[A-8I|\vec{0}] = \left[\begin{array}{cc|c} 2 & -5 & 0 \\ 8 & -20 & 0 \end{array}\right] \xrightarrow[\frac{1}{4}R_2]{} \left[\begin{array}{cc|c} 2 & -5 & 0 \\ 2 & -5 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -5/2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$v_1 = \frac{5}{2} v_2 \Rightarrow \vec{v_1} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

$\lambda=-10$:

$$[A+10I|\vec{0}] = \left[\begin{array}{cc|c} 20 & -5 & 0 \\ 8 & -2 & 0 \end{array}\right] \xrightarrow[\frac{1}{5}R_1]{\frac{1}{2}R_2} \left[\begin{array}{cc|c} 4 & -1 & 0 \\ 4 & -1 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -1/4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$v_1 = \frac{1}{4} v_2 \Rightarrow \vec{v_1} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

=> General Solution :

$$\vec{x} = c_1 \begin{bmatrix} 5 \\ 2 \end{bmatrix} e^{8t} + c_2 \begin{bmatrix} 1 \\ 4 \end{bmatrix} e^{-10t}$$

③ $\vec{x}' = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \vec{x}$

$$\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 3 - 8 = \lambda^2 - 4\lambda - 5 = (\lambda-5)(\lambda+1)$$

$\boxed{\lambda \in \{5, -1\}}$

$\lambda = 5$:

$$[A - 5I \mid \vec{0}] = \left[\begin{array}{cc|c} -4 & 2 & 0 \\ 4 & -2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -1/2 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow v_1 = \tfrac{1}{2} v_2 \Rightarrow \vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$\lambda = -1$:

$$[A + I \mid \vec{0}] = \left[\begin{array}{cc|c} 2 & 2 & 0 \\ 4 & 4 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow v_1 = -v_2 \Rightarrow \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\boxed{\vec{x} = c_1 e^{5t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}}$$

④ $\begin{cases} x'(t) = -4x + 2y \\ y'(t) = -\frac{5}{2}x + 2y \end{cases}$ $\quad A = \begin{pmatrix} -4 & 2 \\ -5/2 & 2 \end{pmatrix}$

$$\det(A-\lambda I) = \begin{vmatrix} -4-\lambda & 2 \\ -5/2 & 2-\lambda \end{vmatrix} = \lambda^2 + 2\lambda - 8 + 5 = \lambda^2 + 2\lambda - 3 = (\lambda+3)(\lambda-1)$$

$\boxed{\lambda \in \{-3, 1\}}$

$\lambda = -3$:

$$\left[\begin{array}{cc|c} -1 & 2 & 0 \\ -5/2 & 5 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow v_1 = 2v_2 \Rightarrow \vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$\lambda = 1$:

$$\left[\begin{array}{cc|c} -5 & 2 & 0 \\ -5/2 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -2/5 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow v_1 = +\tfrac{2}{5} v_2 \Rightarrow \vec{v}_2 = \begin{bmatrix} 2 \\ +5 \end{bmatrix}$$

$$\boxed{\vec{x} = c_1 e^{-3t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{t} \begin{bmatrix} 2 \\ +5 \end{bmatrix}}$$

(5) $\begin{cases} x' = x+y-z \\ y' = 2y \\ z' = y-z \end{cases}$ $\quad A = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & 0 \\ 0 & 1 & -1 \end{pmatrix}$

$$\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 1 & -1 \\ 0 & 2-\lambda & 0 \\ 0 & 1 & -1-\lambda \end{vmatrix} = (1-\lambda)(\lambda-2)(\lambda+1) \Rightarrow \text{eigenvalues: } \lambda \in \{1,-1,2\}$$

$\lambda=1$:

$$[A-I|\vec{0}] = \left[\begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right] \quad \begin{cases} v_2 = v_3 \\ v_2 = 0 \\ v_2 = 2v_3 \end{cases} \Rightarrow v_2 = v_3 = 0,\ v_1 \in \mathbb{R} \Rightarrow \vec{v_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$\lambda=2$:

$$[A-2I|\vec{0}] = \left[\begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -3 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$v_1 = 2v_3$
$v_2 = 3v_3$ $\quad \vec{v_2} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$

$\lambda=-1$:

$$[A+I|\vec{0}] = \left[\begin{array}{ccc|c} 2 & 1 & -1 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 2 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 2 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$v_1 = \frac{1}{2}v_3$
$v_2 = 0$ $\quad \vec{v_3} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

General Solution:

$$\boxed{\vec{X} = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} e^{2t} + c_3 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} e^{-t}}$$

6 $\quad \vec{x}' = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 3 & -1 \end{pmatrix} \vec{x}$

$$\det(A-\lambda I) = \begin{vmatrix} -1-\lambda & 1 & 0 \\ 1 & 2-\lambda & 1 \\ 0 & 3 & -1-\lambda \end{vmatrix} = -(1+\lambda)\left(-(1+\lambda)(2-\lambda)-3\right) + (1+\lambda)$$

$$= (1+\lambda)\left(2+\lambda-\lambda^2+4\right) = (1+\lambda)\left(-\lambda^2+\lambda+6\right) = -(1+\lambda)(\lambda-3)(\lambda+2)$$

$\Rightarrow$ eigenvalues : $\lambda=-1$; $\lambda=3$; $\lambda=-2$

$\lambda=-1$:

$$[A+I|\vec{0}] = \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 3 & 0 & 0 \end{array}\right] \qquad \begin{array}{l} v_2 = 0 \\ v_1 = -v_2 \end{array} \qquad \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$\lambda=3$:

$$[A-3I|\vec{0}] = \left[\begin{array}{ccc|c} -4 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 3 & -4 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -1/4 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -4/3 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -1/4 & 0 & 0 \\ 0 & -3/4 & 1 & 0 \\ 0 & 1 & -4/3 & 0 \end{array}\right]$$

$$\begin{array}{l} v_1 = \frac{1}{4} v_2 \\ v_3 = \frac{3}{4} v_2 \end{array} \qquad \vec{v}_2 = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$$

$\lambda=-2$:

$$[A+2I|\vec{0}] = \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 3 & 1 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 3 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1/3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{array}{l} v_1 = -v_2 \\ v_3 = -3v_2 \end{array} \qquad \vec{v}_3 = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$$

General solution :

$$\boxed{\vec{x} = c_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} e^{3t} + c_3 \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} e^{-2t}}$$

(7) $\vec{x}' = \begin{pmatrix} 1/2 & 0 \\ 1 & -1/2 \end{pmatrix} \vec{x}$

(a). $\det(A-\lambda I) = \begin{vmatrix} 1/2-\lambda & 0 \\ 1 & -1/2-\lambda \end{vmatrix} = (1/2-\lambda)(-1/2-\lambda) \Rightarrow \lambda \in \{-1/2, 1/2\}$

$\lambda = -1/2$

$[A + 1/2 I \mid \vec{0}] = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 1 & 0 & 0 \end{array}\right] \Rightarrow \vec{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\lambda = 1/2$:

$[A - 1/2 I \mid \vec{0}] = \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & -1 & 0 \end{array}\right] \Rightarrow \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

General Solution: $\boxed{c_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-1/2 t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{1/2 t} = \vec{x}(t)}$

(b). $\Phi(t) = \begin{pmatrix} 0 & e^{1/2 t} \\ e^{-1/2 t} & e^{1/2 t} \end{pmatrix}$; $\det \Phi(t) = -1$

$\Phi^{-1}(t) = -\begin{pmatrix} e^{1/2 t} & -e^{1/2 t} \\ -e^{-1/2 t} & 0 \end{pmatrix} = \begin{pmatrix} -e^{1/2 t} & e^{1/2 t} \\ e^{-1/2 t} & 0 \end{pmatrix}$

(c). $\vec{x}(0) = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$

$\Rightarrow \vec{c} = \Phi^{-1}(0) \begin{pmatrix} 3 \\ 5 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$$\boxed{\vec{x} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} e^{-1/2 t} + \begin{bmatrix} 3 \\ 3 \end{bmatrix} e^{1/2 t}}$$