Math 2552 - Differential Equations **Homework 19**
Sections F1 – F4; L1 – L4
Georgia Institute of Technology, Fall 2015

First-Order Linear Systems of Differential Equations with Constant Coefficients
I. Real, Distinct Eigenvalues

Solve the following systems:

1. $$\begin{cases} \frac{dx}{dt} = -4x + y + z \\ \frac{dy}{dt} = x + 5y - z \\ \frac{dz}{dt} = y - 3z. \end{cases}$$

2. $$\mathbf{x}' = \begin{pmatrix} 10 & -5 \\ 8 & -12 \end{pmatrix} \mathbf{x}.$$

3. $$\mathbf{x}' = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \mathbf{x}.$$

4. $$\begin{cases} x'(t) = -4x + 2y \\ y'(t) = -\frac{5}{2}x + 2y. \end{cases}$$

5. $$\begin{cases} x'(t) = x + y - z \\ y'(t) = 2y \\ z'(t) = y - z. \end{cases}$$

6. $$\mathbf{x}' = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 3 & -1 \end{pmatrix} \mathbf{x}.$$

7. Consider the system:

$$\mathbf{x}' = \begin{pmatrix} 1/2 & 0 \\ 1 & -1/2 \end{pmatrix} \mathbf{x}.$$

(a). Find the general solution.
(b). Find a fundamental matrix $\Phi(t)$ and find its inverse $\Phi^{-1}(t)$.
(c). Use the information in part (b) to find the solution subject to the initial condition:

$$\mathbf{x}(0) = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$$

Eigenvalues & Eigenvectors :

Def.: Given an $n\times n$ matrix A, a number $\lambda \in \mathbb{C}$ is called an eigenvalue of A if there is a non-zero vector $\vec{v}$ such that $\boxed{A\vec{v} = \lambda\vec{v}}$

Any such vector $\vec{v}$ is called an eigenvector for λ.

We may write

$$A\vec{v} = \lambda\vec{v} \iff A\vec{v} - \lambda\vec{v} = \vec{0} \iff (A-\lambda I)\vec{v} = \vec{0}$$

This equation has non-trivial solutions if and only if $\boxed{\det(A-\lambda I) = 0}$

Characteristic Equation of A

=> The eigenvalues of A are the roots of the characteristic equation!

Example: $A = \begin{bmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{bmatrix}$

$$\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 2 & 1 \\ 6 & -1-\lambda & 0 \\ -1 & -2 & -1-\lambda \end{vmatrix} = \begin{vmatrix} 6 & -1-\lambda \\ -1 & -2 \end{vmatrix} - (1+\lambda)\begin{vmatrix} 1-\lambda & 2 \\ 6 & -1-\lambda \end{vmatrix}$$

$$= -12 - 1 - \lambda - (1+\lambda)(-1+\lambda^2-12)$$

$$= -13 - \lambda + 1 - \lambda^2 + 12 + \lambda - \lambda^3 + 12\lambda = -\lambda^3 - \lambda^2 + 12\lambda$$

Char. Eqn.: $\boxed{-\lambda^3 - \lambda^2 + 12\lambda = 0}$

Roots: $\lambda(\lambda^2+\lambda-12) = 0 \Rightarrow \lambda(\lambda-3)(\lambda+4) = 0 \Rightarrow \boxed{\lambda \in \{0, 3, -4\}}$ ← eigenvalues

To find the eigenvectors corresponding to an eigenvalue λ, we must solve the linear system $(A-\lambda I)\vec{v} = \vec{0}$. So, reduce $[A-\lambda I \,|\, \vec{0}]$:

$\lambda = 0$:

$$[A\,|\,\vec{0}] = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 6 & -1 & 0 & 0 \\ -1 & -2 & -1 & 0 \end{array}\right] \xrightarrow[R_1+R_2]{-6R_1+R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -13 & -6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[-\frac{1}{13}R_2]{} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 6/13 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow[-2R_2+R_1]{} \left[\begin{array}{ccc|c} 1 & 0 & 1/13 & 0 \\ 0 & 1 & 6/13 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow$ System becomes $\begin{cases} v_1 + \frac{1}{13} v_3 = 0 \\ v_2 + \frac{6}{13} v_3 = 0 \end{cases}$ $\begin{cases} v_1 = -\frac{1}{13} v_3 \\ v_2 = -\frac{6}{13} v_3 \end{cases}$

$\Rightarrow$ any vector of the form: $\begin{bmatrix} -\frac{1}{13}c \\ -\frac{6}{13}c \\ c \end{bmatrix}$

is an eigenvector for $\lambda = 0$. Choose $c = -13$: $\boxed{\vec{v}_1 = \begin{bmatrix} 1 \\ 6 \\ -13 \end{bmatrix}}$

$\lambda_2 = -4$

$$[A + 4I \mid \vec{0}] = \left[\begin{array}{ccc|c} 5 & 2 & 1 & 0 \\ 6 & 3 & 0 & 0 \\ -1 & -2 & 3 & 0 \end{array}\right] \xrightarrow[R_{3 \leftrightarrow 1}]{-R_3} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 6 & 3 & 0 & 0 \\ 5 & 2 & 1 & 0 \end{array}\right] \xrightarrow[-5R_1+R_3]{-6R_1+R_2} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & -9 & +18 & 0 \\ 0 & -8 & 16 & 0 \end{array}\right]$$

$$\xrightarrow[\substack{-1/9 R_2 \\ -1/8 R_3}]{} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right] \xrightarrow[R_3 - R_2]{} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[R_1 - 2R_2]{} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow$ any vector of the form $\begin{bmatrix} -c \\ 2c \\ c \end{bmatrix}$ is an eigenvector for $\lambda = -4$

Choose $c = 1 \Rightarrow \boxed{\vec{v}_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}}$

$\lambda_3 = 3$

$$[A - 3I \mid \vec{0}] = \left[\begin{array}{ccc|c} -2 & 2 & 1 & 0 \\ 6 & -4 & 0 & 0 \\ -1 & -2 & -4 & 0 \end{array}\right] \xrightarrow[\substack{R_{3 \leftrightarrow 1} \\ -R_3}]{-R_3} \left[\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 6 & -4 & 0 & 0 \\ 2 & -2 & -1 & 0 \end{array}\right] \xrightarrow[-2R_1+R_3]{-6R_1+R_2} \left[\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 0 & -16 & -24 & 0 \\ 0 & -6 & -9 & 0 \end{array}\right]$$

$$\xrightarrow[\substack{-1/16 R_2 \\ -1/9 R_3}]{} \left[\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 0 & 1 & 3/2 & 0 \\ 0 & 1 & 3/2 & 0 \end{array}\right] \xrightarrow[R_3 - R_2]{} \left[\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 0 & 1 & 3/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[R_1 - 2R_2]{} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 3/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow$ any vector of the form $\begin{bmatrix} -c \\ -\frac{3}{2}c \\ c \end{bmatrix}$ is an eigenvector for $\lambda = 3$

Choose $c = -2 \Rightarrow \boxed{\vec{v}_3 = \begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix}}$

Homogeneous Linear Systems with Constant Coefficients

Consider a homogeneous linear n-dimensional system:

$$\vec{x}' = A\vec{x}$$

where A is an n×n matrix of constants. Look for a solution of the form

$$\vec{x} = \vec{v}e^{\lambda t} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} e^{\lambda t}$$

where $\vec{v}$ is a vector of constants. If $\vec{x} = \vec{v}e^{\lambda t}$, then $\vec{x}' = \lambda\vec{v}e^{\lambda t}$, so

$$A\vec{x} - \vec{x}' = \vec{0} \iff A\vec{v}e^{\lambda t} - \lambda\vec{v}e^{\lambda t} = \vec{0} \iff (A\vec{v} - \lambda\vec{v})e^{\lambda t} = \vec{0}$$

$$\iff \boxed{A\vec{v} = \lambda\vec{v}}$$

$\vec{v}$ is an eigenvector of A corresponding to the eigenvalue λ.

Recall: When an n×n matrix has n distinct eigenvalues $\lambda_1, .., \lambda_n$, then a set of n linearly independent eigenvectors $\vec{v_1}, .., \vec{v_n}$ can always be found.

In this case, $\vec{x_1} = \vec{v_1}e^{\lambda_1 t}$; $\vec{x_2} = \vec{v_2}e^{\lambda_2 t}$; ...; $\vec{x_n} = \vec{v_n}e^{\lambda_n t}$

are n linearly independent solutions to the system => fundamental set.

=> General Solution to Homogeneous Systems:

Case 1: The matrix A has n distinct real eigenvalues

Consider the homogeneous linear system of dimension n with constant coefficients: $\vec{x}' = A\vec{x}$.

If the matrix A has n distinct real eigenvalues $\lambda_1, \lambda_2, .., \lambda_n$ with corresponding eigenvectors $\vec{v_1}, \vec{v_2}, .., \vec{v_n}$, then the general solution to the system on $\mathbb{R}$ is:

$$\boxed{\vec{x} = c_1\vec{v_1}e^{\lambda_1 t} + c_2\vec{v_2}e^{\lambda_2 t} + \ldots + c_n\vec{v_n}e^{\lambda_n t}}$$

Example: $\begin{cases} \dfrac{dx}{dt} = 2x + 3y \\ \dfrac{dy}{dt} = 2x + y \end{cases}$ $\qquad A = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}$

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 3 \\ 2 & 1-\lambda \end{vmatrix} = (2-\lambda)(1-\lambda) - 6 = \lambda^2 - 3\lambda - 4 = (\lambda - 4)(\lambda + 1)$$

Eigenvalues: $\boxed{\lambda \in \{4, -1\}}$

$\lambda = 4$: $\left[A - 4I \mid \vec{0}\right] = \left[\begin{array}{cc|c} -2 & 3 & 0 \\ 2 & -3 & 0 \end{array}\right] \xrightarrow{-R_1} \left[\begin{array}{cc|c} 2 & -3 & 0 \\ 2 & -3 & 0 \end{array}\right] \xrightarrow{R_2 - R_1} \left[\begin{array}{cc|c} 2 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$\Rightarrow 2v_1 - 3v_2 = 0 \Rightarrow v_1 = \frac{3}{2} v_2 \Rightarrow \vec{v}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$\lambda = -1$: $\left[A + I \mid \vec{0}\right] = \left[\begin{array}{cc|c} 3 & 3 & 0 \\ 2 & 2 & 0 \end{array}\right] \xrightarrow[\frac{1}{2}R_2]{\frac{1}{3}R_1} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right] \xrightarrow{R_2 - R_1} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$\Rightarrow v_1 + v_2 = 0 \Rightarrow v_1 = -v_2 \Rightarrow \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\Rightarrow$ General Solution: $\boxed{\vec{x} = c_1 \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{4t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}}$