Homework (18)
-Solutions-

(1) $\begin{cases} \dfrac{dx}{dt} = 3x - 5y \\ \dfrac{dy}{dt} = 4x + 8y \end{cases}$ $\quad \vec{x}' = P\vec{x}$, where $\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$; $P = \begin{pmatrix} 3 & -5 \\ 4 & 8 \end{pmatrix}$

(2) $\begin{cases} \dfrac{dx}{dt} = x - y + z + t - 1 \\ \dfrac{dy}{dt} = 2x + y - z - 3t^2 \\ \dfrac{dz}{dt} = x + y + z + t^2 - t + 2 \end{cases}$ $\quad \vec{x}' = P\vec{x} + g(t)$, where $\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$,

$P = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix}$; $g(t) = \begin{pmatrix} t-1 \\ -3t^2 \\ t^2 - t + 2 \end{pmatrix}$.

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(3) $\vec{x}' = \begin{pmatrix} 4 & 2 \\ -1 & 3 \end{pmatrix}\vec{x} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t$ $\quad \begin{pmatrix} 4 & 2 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4x_1 + 2x_2 \\ -x_1 + 3x_2 \end{pmatrix}$

$$\begin{cases} x_1' = 4x_1 + 2x_2 + e^t \\ x_2' = -x_1 + 3x_2 - e^t \end{cases}$$

(4) $\dfrac{d}{dt}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & -1 & 2 \\ 3 & -4 & 1 \\ -2 & 5 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} e^{-t} - \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} t$

$$\begin{cases} x' = x - y + 2z + e^{-t} - 3t \\ y' = 3x - 4y + z + 2e^{-t} + t \\ z' = -2x + 5y + 6z + 2e^{-t} - t. \end{cases}$$

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(5) $\begin{cases} x' = 3x - 4y \\ y' = 4x - 7y \end{cases}$; $\vec{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-5t}$

$\vec{x}' = \begin{pmatrix} 3 & -4 \\ 4 & -7 \end{pmatrix}\vec{x}$ $\quad \vec{x}' = -5\begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-5t} = \begin{pmatrix} -5 \\ -10 \end{pmatrix} e^{-5t}$ ✓

$$\begin{pmatrix} 3 & -4 \\ 4 & -7 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-5t} = \begin{pmatrix} -5 \\ -10 \end{pmatrix} e^{-5t}$$

6 $\vec{x}\,' = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ -2 & 0 & -1 \end{pmatrix} \vec{x}$; $\vec{x} = \begin{pmatrix} \sin t \\ -\frac{1}{2}\sin t - \frac{1}{2}\cos t \\ -\sin t + \cos t \end{pmatrix}$

$\vec{x}\,' = \begin{pmatrix} \cos t \\ -\frac{1}{2}\cos t + \frac{1}{2}\sin t \\ -\cos t - \sin t \end{pmatrix}$;

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ -2 & 0 & -1 \end{pmatrix}\begin{pmatrix} \sin t \\ -1/2 \sin t - 1/2 \cos t \\ -\sin t + \cos t \end{pmatrix} = \begin{pmatrix} \cos t \\ 1/2 \sin t - 1/2 \cos t \\ -\sin t - \cos t \end{pmatrix}$$

✓

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7 $\vec{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t}$; $\vec{x}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-6t}$

$W(\vec{x}_1, \vec{x}_2) = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} e^{-8t} = -2e^{-8t} \neq 0 \Rightarrow$ fundamental set on $\mathbb{R}$. ✓

8 $\vec{x}_1 = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$; $\vec{x}_2 = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$; $\vec{x}_3 = \begin{pmatrix} 3 \\ -6 \\ 12 \end{pmatrix} + t\begin{pmatrix} 2 \\ 4 \\ 4 \end{pmatrix}$ ✗

Not linearly independent: $\vec{x}_3 = 3\vec{x}_1 - t\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = 3\vec{x}_1 - (\vec{x}_1 - \vec{x}_2) = 2\vec{x}_1 + \vec{x}_2$.

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9 $\vec{x}\,' = \begin{pmatrix} 4 & 1 \\ 6 & 5 \end{pmatrix} \vec{x}$; $\vec{x}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{2t}$; $\vec{x}_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{7t}$

Fundamental Matrix: $\Phi(t) = \begin{pmatrix} e^{2t} & e^{7t} \\ -2e^{2t} & 3e^{7t} \end{pmatrix}$

Inverse: $\det \Phi(t) = 5e^{9t} \Rightarrow \Phi^{-1}(t) = \frac{1}{5e^{9t}}\begin{pmatrix} 3e^{7t} & -e^{7t} \\ 2e^{2t} & e^{2t} \end{pmatrix}$.

Special Matrix $\Psi(0) = I$:

$$\Psi(t) = \Phi(t)\,\Phi^{-1}(0) = \begin{pmatrix} e^{2t} & e^{7t} \\ -2e^{2t} & 3e^{7t} \end{pmatrix} \frac{1}{5}\begin{pmatrix} 3 & -1 \\ 2 & 1 \end{pmatrix}$$

$$= \frac{1}{5}\begin{pmatrix} 3e^{2t} + 2e^{7t} & -e^{2t} + e^{7t} \\ -6e^{2t} + 6e^{7t} & 2e^{2t} + 3e^{7t} \end{pmatrix}.$$

10 $\vec{x}' = \begin{pmatrix} 3 & -2 \\ 5 & -3 \end{pmatrix} \vec{x}$; $\vec{x}_1 = \begin{pmatrix} 2\cos t \\ 3\cos t + \sin t \end{pmatrix}$; $\vec{x}_2 = \begin{pmatrix} -2\sin t \\ \cos t - 3\sin t \end{pmatrix}$

$$\Phi(t) = \begin{pmatrix} 2\cos t & -2\sin t \\ 3\cos t + \sin t & \cos t - 3\sin t \end{pmatrix}$$

$$\det \Phi(t) = 2\cos^2 t - 6\sin t\cos t + 6\sin t\cos t + 2\sin^2 t = \underline{\underline{2}}$$

$$\Rightarrow \quad \Phi^{-1}(t) = \frac{1}{2}\begin{pmatrix} \cos t - 3\sin t & 2\sin t \\ -3\cos t - \sin t & 2\cos t \end{pmatrix}$$

Special Matrix with $\Psi(\pi/2) = I$:

$$\Psi(t) = \Phi(t)\,\Phi^{-1}(\pi/2) = \begin{pmatrix} 2\cos t & -2\sin t \\ 3\cos t + \sin t & \cos t - 3\sin t \end{pmatrix} \frac{1}{2}\begin{pmatrix} -3 & 2 \\ -1 & 0 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} -6\cos t + 2\sin t & 4\cos t \\ \begin{matrix} -9\cos t - 3\sin t \\ -\cos t + 3\sin t \end{matrix} & 6\cos t + 2\sin t \end{pmatrix}$$

$$= \begin{pmatrix} -3\cos t + \sin t & 2\cos t \\ -5\cos t & 3\cos t + \sin t \end{pmatrix}$$