Math 2552 - Differential Equations **Homework 18**
Sections F1 – F4; L1 – L4
Georgia Institute of Technology, Fall 2015

Systems of Differential Equations
Preliminary Theory and Definitions

Write the following systems in matrix form:

1. $\begin{cases} \frac{dx}{dt} = 3x - 5y \\ \frac{dy}{dt} = 4x + 8y \end{cases}$

2. $\begin{cases} \frac{dx}{dt} = x - y + z + t - 1 \\ \frac{dy}{dt} = 2x + y - z - 3t^2 \\ \frac{dz}{dt} = x + y + z + t^2 - t + 2 \end{cases}$

Write the following systems without using matrices:

3. $\mathbf{x}' = \begin{pmatrix} 4 & 2 \\ -1 & 3 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t.$

4. $\dfrac{d}{dt}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & -1 & 2 \\ 3 & -4 & 1 \\ -2 & 5 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} e^{-t} - \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} t.$

Verify that the given $\mathbf{x}$ is a solution to the system:

5. $\begin{cases} x' = 3x - 4y \\ y' = 4x - 7y \end{cases}$; $\mathbf{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-5t}.$

6. $\mathbf{x}' = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ -2 & 0 & -1 \end{pmatrix} \mathbf{x}$; $\mathbf{x} = \begin{pmatrix} \sin t \\ -1/2 \sin t - 1/2 \cos t \\ -\sin t + \cos t \end{pmatrix}$

The following vectors are sets of solutions to a homogeneous linear system $\mathbf{x}' = \mathbf{P}\mathbf{x}$. Check whether or not they are fundamental sets.

7. $\mathbf{x_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t}$; $\mathbf{x_2} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-6t}.$

8. $\mathbf{x_1} = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$; $\mathbf{x_2} = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$; $\mathbf{x_3} = \begin{pmatrix} 3 \\ -6 \\ 12 \end{pmatrix} + t \begin{pmatrix} 2 \\ 4 \\ 4 \end{pmatrix}$

9. Below is a system and a fundamental set of solutions:

$$\mathbf{x}' = \begin{pmatrix} 4 & 1 \\ 6 & 5 \end{pmatrix} \mathbf{x};\ \mathbf{x_1} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{2t};\ \mathbf{x_2} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{7t}.$$

Write the fundamental matrix $\mathbf{\Phi}(t)$ and find its inverse $\mathbf{\Phi}^{-1}(t)$. Find the special fundamental matrix that satisfies $\mathbf{\Psi}(0) = \mathbf{I}$.

10. Below is a system and a fundamental set of solutions:

$$\mathbf{x}' = \begin{pmatrix} 3 & -2 \\ 5 & -3 \end{pmatrix} \mathbf{x};\ \mathbf{x_1} = \begin{pmatrix} 2\cos t \\ 3\cos t + \sin t \end{pmatrix};\ \mathbf{x_2} = \begin{pmatrix} -2\sin t \\ \cos t - 3\sin t \end{pmatrix}.$$

Write the fundamental matrix $\mathbf{\Phi}(t)$ and find its inverse $\mathbf{\Phi}^{-1}(t)$. Find the special fundamental matrix that satisfies $\mathbf{\Psi}(\pi/2) = \mathbf{I}$.

First Order Linear Systems

General form (Canonical / normal form - the coefficients of x_i' are 1):

$$\begin{cases} x_1' = p_{11}(t)x_1 + p_{12}(t)x_2 + \dots + p_{1n}(t)x_n + g_1(t) \\ x_2' = p_{21}(t)x_1 + p_{22}(t)x_2 + \dots + p_{2n}(t)x_n + g_2(t) \\ \vdots \qquad\qquad\qquad\qquad \vdots \qquad \vdots \\ x_n' = p_{n1}(t)x_1 + p_{n2}(t)x_2 + \dots + p_{nn}(t)x_n + g_n(t) \end{cases} \qquad t \in I$$

In matrix notation:

$$\boxed{\vec{x}' = P(t)\vec{x} + \vec{g}(t)}$$

where:

$$\vec{x} = \vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} \quad ; \qquad \vec{g}(t) = \begin{bmatrix} g_1(t) \\ g_2(t) \\ \vdots \\ g_n(t) \end{bmatrix}$$

vector of unknown functions $x_1, \dots, x_n$ of the variable t

the input or forcing function; determines homogeneity of the system:

- if $\vec{g}(t) = \vec{0}$ for all $t \in I$, we say the system is homogeneous
- otherwise, the system is said to be non-homogeneous.

$$P(t) = \begin{bmatrix} p_{11}(t) & \dots & p_{1n}(t) \\ \vdots & & \vdots \\ p_{n1}(t) & \dots & p_{nn}(t) \end{bmatrix}$$

the matrix of coefficients

- A solution is a vector $\vec{x}(t)$ with n component functions, differentiable at all $t \in I$, that satisfies the system.
- If we are also given an initial condition: $\boxed{\vec{x}(t_0) = \vec{x}_0}$ at some $t_0 \in I$, where $\vec{x}_0$ is a constant vector, we have an initial value problem.

Transforming an ODE into a System :

Any n^{th} order linear ODE can be written as a first order linear system :

$$\boxed{\frac{d^n y}{dt^n} + p_1(t)\frac{d^{n-1}y}{dt^{n-1}} + \ldots + p_{n-1}(t)\frac{dy}{dt} + p_n(t)y = g(t)}$$

Introduce n variables :

$$\boxed{x_1 = y \ ; \ x_2 = y' \ ; \ x_3 = y'' \ ; \ \ldots \ ; \ x_n = y^{(n-1)}}$$

$$\Rightarrow \begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ x_3' = x_4 \\ \vdots \\ x_{n-1}' = x_n \\ x_n' = -p_1(t)x_n - \ldots - p_{n-1}(t)x_2 - p_n(t)x_1 + g(t) \end{cases}$$

$$\Rightarrow \boxed{\vec{x}' = P(t)\vec{x} + \vec{g}(t)} \quad \text{where} \quad P(t) = \begin{bmatrix} 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & 0 & \ldots & 1 \\ -p_n(t) & -p_{n-1}(t) & \ldots & & & -p_1(t) \end{bmatrix} \ ; \ \vec{g}(t) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ g(t) \end{bmatrix}$$

Example : $\boxed{2y''' - 6y'' + 4y' + y = \sin t}$

$$y''' = 3y'' - 2y' - \frac{1}{2}y + \frac{1}{2}\sin t$$

$$x_1 = y \ ; \ x_2 = y' \ ; \ x_3 = y''$$

$$\begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ x_3' = 3x_3 - 2x_2 - \frac{1}{2}x_1 + \sin t \end{cases}$$

$$\vec{x}' = P(t)\vec{x} + \vec{g}(t) \ , \text{where } P(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1/2 & -2 & 3 \end{bmatrix} \ ; \ \vec{g}(t) = \begin{bmatrix} 0 \\ 0 \\ \sin t \end{bmatrix}$$

Existence & Uniqueness :

Consider the initial value problem:

$$\vec{x}' = P(t)\vec{x} + \vec{g}(t) \quad ; \quad \vec{x}(t_0) = \vec{x}_0 \qquad (*)$$

where $t \in I$ for some open interval I. If $P(t)$ and $g(t)$ are continuous on I (i.e. every component function is continuous on I), then $(*)$ has a unique solution $\vec{x}(t)$ on I.

Superposition :

If $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k$ are solutions to the homogeneous linear system $\boxed{\vec{x}' = P(t)\vec{x}}$ on an interval I, then any linear combination $c_1\vec{x}_1 + c_2\vec{x}_2 + \ldots + c_k\vec{x}_k$ is also a solution to the system on I.

Def.: Linear Independence

We say that n vector functions $\vec{x}_1(t), \ldots, \vec{x}_n(t)$ are linearly independent on an interval I provided that

$$\boxed{c_1\vec{x}_1(t) + \ldots + c_n\vec{x}_n(t) = 0, \ \forall t \in I} \iff \boxed{c_1 = c_2 = \ldots = c_n = 0}$$

Otherwise, if there exist constants $c_1, c_2, \ldots, c_n$ not all zero such that

$$c_1\vec{x}_1(t) + \ldots + c_n\vec{x}_n(t) = 0, \ \forall t \in I$$

we say $\vec{x}_1, \ldots, \vec{x}_n$ are linearly dependent on I.

Linear Independence & Wronskian :

Suppose that :

$$\vec{x}_1 = \begin{bmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{bmatrix}, \quad \vec{x}_2 = \begin{bmatrix} x_{12} \\ x_{22} \\ \vdots \\ x_{n2} \end{bmatrix}, \ldots, \quad \vec{x}_n = \begin{bmatrix} x_{1n} \\ x_{2n} \\ \vdots \\ x_{nn} \end{bmatrix}$$

are n solutions to the homogeneous system $\vec{x}' = P(t)\vec{x}$ on some interval I. Then the set $\{\vec{x}_1, \ldots, \vec{x}_n\}$ of solutions is linearly independent if and only if the Wronskian :

$$W(\vec{x}_1, \ldots, \vec{x}_n) := \begin{vmatrix} x_{11} & x_{12} & \ldots & x_{1n} \\ x_{21} & x_{22} & \ldots & x_{2n} \\ \vdots & & & \vdots \\ x_{n1} & x_{n2} & \ldots & x_{nn} \end{vmatrix}$$

satisfies $W(\vec{x}_1, \ldots, \vec{x}_n)(t) \neq 0$, $\forall t \in I$.

Def.: Fundamental Set of Solutions / Fundamental Matrix

Consider the homogeneous linear system $\vec{x}' = P(t)\vec{x}$, $t \in I$. A fundamental set of solutions to this system on I is any set $\{\vec{x}_1, \dots, \vec{x}_n\}$ of n linearly independent solutions (where n is the dimension of the system). Given a fundamental set:

$$\vec{x}_1 = \begin{bmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{bmatrix}, \quad \vec{x}_2 = \begin{bmatrix} x_{12} \\ x_{22} \\ \vdots \\ x_{n2} \end{bmatrix}, \quad \dots, \quad \vec{x}_n = \begin{bmatrix} x_{1n} \\ x_{2n} \\ \vdots \\ x_{nn} \end{bmatrix}$$

the matrix:

$$\Phi(t) := \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{bmatrix}$$

is called a fundamental matrix of the system on I.

Def.: General Solution:

Given a fundamental set $\{\vec{x}_1, \dots, \vec{x}_n\}$ of solutions to the homogeneous system $\vec{x}' = P(t)\vec{x}$, $t \in I$, the general solution to the system is

$$\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2 + \dots + c_n\vec{x}_n = \Phi(t)\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \Phi(t)\vec{c}$$

where $c_1, c_2, \dots, c_n$ are arbitrary constants.

Remark: If $\Phi(t)$ is the fundamental matrix corresponding to a fundamental set $\{\vec{x}_1, \dots, \vec{x}_n\}$ of solutions:

$$\boxed{\det(\Phi(t)) = W(\vec{x}_1, \dots, \vec{x}_n)}$$

By linear independence, $W(\vec{x}_1, \dots, \vec{x}_n)(t) \neq 0$ for all $t \in I$, so $\det(\Phi(t)) \neq 0$, $\forall t \in I$. In other words:

> The fundamental matrix $\Phi(t)$ of a homogeneous linear system is invertible for all $t \in I$.

Remark: It can be shown that if $\{\vec{x}_1, \dots, \vec{x}_n\}$ are any n solutions to $\vec{x}' = P(t)\vec{x}$, then either $W(\vec{x}_1, \dots, \vec{x}_n)(t) = 0$ for all $t \in I$, or $W(\vec{x}_1, \dots, \vec{x}_n)(t) \neq 0$ for all $t \in I$. So if we can show $W(\vec{x}_1, \dots, \vec{x}_n)(t_0) \neq 0$ for some $t_0 \in I$, then $W(\vec{x}_1, \dots, \vec{x}_n)(t) \neq 0$, for all $t \in I$.

Remark: Suppose we have an IVP: $\vec{x}' = P(t)\vec{x}$; $\vec{x}(t_0) = \vec{x}_0$ on some interval I.

The general solution:

$$\vec{x}(t) = \Phi(t)\vec{c},$$

where $\Phi(t)$ is a fundamental matrix and $\vec{c}$ is a vector of arbitrary constants, must then satisfy:

$$\vec{x}(t_0) = \Phi(t_0)\vec{c} = \vec{x}_0 \Rightarrow \boxed{\vec{c} = \Phi^{-1}(t_0)\vec{x}_0}$$

So the solution to the IVP is given by:

$$\boxed{\vec{x}(t) = \Phi(t)\Phi^{-1}(t_0)\vec{x}_0}$$

Def.: Special Fundamental Matrix:

Let $\vec{x}' = P(t)\vec{x}$ be an n-dimensional linear homogeneous system. Suppose that $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are solutions to this system satisfying

$$\vec{v}_1(t_0) = \vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} ; \quad \vec{v}_2(t_0) = \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} ; \ \ldots \ ; \quad \vec{v}_n(t_0) = \vec{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

for some fixed $t_0 \in I$. The special fundamental matrix $\Psi(t)$ is the matrix whose column vectors are $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$.

- $\Psi(t_0) = I_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$

This also shows that any homogeneous system $\vec{x}' = P(t)\vec{x}$ has a fundamental set on I.

- $\Psi(t)$ is a fundamental matrix (i.e. $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is a fundamental set).

  Since $\det(\Psi(t_0)) = \det(I_n) = 1 \neq 0$, we have that

  $$\det(\Psi(t_0)) = W(\vec{x}_1, \ldots, \vec{x}_n)(t_0) \neq 0$$

  So the Wronskian is non-zero for some $t_0 \in I$, therefore it is non-zero everywhere on I.

- $\boxed{\Psi(t) = \Phi(t)\Phi^{-1}(t_0)}$

$$\left.\begin{aligned} \vec{v}_1(t) &= \Phi(t)\Phi^{-1}(t_0)\vec{e}_1 \\ \vec{v}_2(t) &= \Phi(t)\Phi^{-1}(t_0)\vec{e}_2 \\ &\vdots \\ \vec{v}_n(t) &= \Phi(t)\Phi^{-1}(t_0)\vec{e}_n \end{aligned}\right\} \Rightarrow$$ The columns of $\Phi(t)\Phi^{-1}(t_0)$ are the vectors $\vec{v}_1, \ldots, \vec{v}_n$, so $\Phi(t)\Phi^{-1}(t_0) = \Psi(t)$.

- In terms of $\Psi(t)$, the solution to the IVP $\boxed{\vec{x}' = P(t)\vec{x} \ ; \ \vec{x}(t_0) = \vec{x}_0}$ is $\boxed{\vec{x}(t) = \Psi(t)\vec{x}_0}$

Example: Consider the system $\vec{x}' = \begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix} \vec{x}$ on $\mathbb{R}$.

- Check that $\vec{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-2t}$ and $\vec{x}_2 = \begin{bmatrix} 3 \\ 5 \end{bmatrix} e^{6t}$ are solutions:

$$\vec{x}_1' = -2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-2t} = \begin{bmatrix} -2 \\ 2 \end{bmatrix} e^{-2t} \quad \checkmark$$

$$\begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix} \vec{x}_1 = \begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-2t} = \begin{bmatrix} -2 \\ 2 \end{bmatrix} e^{-2t}$$

$$\vec{x}_2' = 6 \begin{bmatrix} 3 \\ 5 \end{bmatrix} e^{6t} = \begin{bmatrix} 18 \\ 30 \end{bmatrix} e^{6t} \quad \checkmark$$

$$\begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix} \vec{x}_2 = \begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix} e^{6t} = \begin{bmatrix} 18 \\ 30 \end{bmatrix} e^{6t}$$

- Check that $\{\vec{x}_1, \vec{x}_2\}$ is a <u>fundamental set</u>:

$$W(\vec{x}_1, \vec{x}_2)(t) = \begin{vmatrix} 1 & 3 \\ -1 & 5 \end{vmatrix} e^{4t} = 8e^{4t} \neq 0, \ \forall t \in \mathbb{R}$$

- <u>General Solution</u>: $\vec{x} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 3 \\ 5 \end{bmatrix} e^{6t} = \Phi(t)\vec{c}$

- <u>Fundamental Matrix</u>: $\boxed{\Phi(t) = \begin{bmatrix} e^{-2t} & 3e^{6t} \\ -e^{-2t} & 5e^{6t} \end{bmatrix}}$

- <u>Inverse of Fundamental Matrix</u>:

$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ invertible

$\Rightarrow A^{-1} = \dfrac{1}{\det A} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$

$$\det \Phi(t) = +8e^{4t}$$

$$\Rightarrow \Phi^{-1}(t) = \frac{1}{8e^{4t}} \begin{bmatrix} 5e^{6t} & -3e^{6t} \\ e^{-2t} & e^{-2t} \end{bmatrix}$$

$$\Rightarrow \Phi^{-1}(t) = \begin{bmatrix} 5/8\, e^{2t} & -3/8\, e^{2t} \\ 1/8\, e^{-6t} & 1/8\, e^{-6t} \end{bmatrix}.$$

• Find a special fundamental matrix:

Choose $t_0 = 0$, for example. Then

$$\Phi^{-1}(0) = \begin{bmatrix} 5/8 & -3/8 \\ 1/8 & 1/8 \end{bmatrix}$$

$$\Rightarrow \Psi(t) = \Phi(t)\,\Phi^{-1}(0) = \begin{bmatrix} e^{-2t} & 3e^{6t} \\ -e^{-2t} & 5e^{6t} \end{bmatrix} \begin{bmatrix} 5/8 & -3/8 \\ 1/8 & 1/8 \end{bmatrix}$$

$$\boxed{\Psi(t) = \begin{bmatrix} 5/8\, e^{-2t} + 3/8\, e^{6t} & -3/8\, e^{-2t} + 3/8\, e^{6t} \\ -5/8\, e^{-2t} + 5/8\, e^{6t} & 3/8\, e^{-2t} + 5/8\, e^{6t} \end{bmatrix}}$$

◎ Remark: $\Psi(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2.$

◎ Remark: $\vec{V}_1(t) = \begin{bmatrix} 5/8\, e^{-2t} + 3/8\, e^{6t} \\ -5/8\, e^{-2t} + 5/8\, e^{6t} \end{bmatrix}$ is the solution to the system satisfying $\vec{V}_1(0) = \vec{e}_1$.

$\vec{V}_2(t) = \begin{bmatrix} -3/8\, e^{-2t} + 3/8\, e^{6t} \\ 3/8\, e^{-2t} + 5/8\, e^{6t} \end{bmatrix}$ is the solution to the system satisfying $\vec{V}_2(0) = \vec{e}_2$.

◎ Remark: One may choose a different value of t_0 and obtain a different special fundamental matrix. For example, for $t_0 = 1$:

$$\Phi^{-1}(1) = \begin{bmatrix} 5/8\, e^{2} & -3/8\, e^{2} \\ 1/8\, e^{-6} & 1/8\, e^{-6} \end{bmatrix}$$

$$\Rightarrow \Psi(t) = \Phi(t)\,\Phi^{-1}(1) = \begin{bmatrix} e^{-2t} & 3e^{6t} \\ -e^{-2t} & 5e^{6t} \end{bmatrix} \begin{bmatrix} 5/8\, e^{2} & -3/8\, e^{2} \\ 1/8\, e^{-6} & 1/8\, e^{-6} \end{bmatrix} =$$

$$= \begin{bmatrix} 5/8\, e^{-2t+2} + 3/8\, e^{6t-6} & -3/8\, e^{-2t+2} + 3/8\, e^{6t-6} \\ -5/8\, e^{-2t+2} + 5/8\, e^{6t-6} & 3/8\, e^{-2t+2} + 5/8\, e^{6t-6} \end{bmatrix}$$

Then $\Psi(1) = I_2$, and the first column of $\Psi(t)$ is the solution to the system satisfying $\vec{V}_1(1) = \vec{e}_1$; the second column of $\Psi(t)$ is the solution to the system satisfying $\vec{V}_2(1) = \vec{e}_2$.