Solving Systems of ODEs with the Laplace Transform

1. $$\begin{cases} \dfrac{dx}{dt} = -x+y \\ \dfrac{dy}{dt} = 2x \end{cases}$$

$x(0)=0;\ y(0)=1.$

$$\begin{cases} sX(s) = -X(s) + Y(s) \\ sY(s) - 1 = 2X(s) \end{cases}$$

$$\begin{cases} (s+1)X(s) - Y(s) = 0 \\ -2X(s) + sY(s) = 1 \end{cases}$$

$$-2Y(s) + s(s+1)Y(s) = s+1$$

$$Y(s)(s^2+s-2) = s+1 \Rightarrow \boxed{Y(s) = \frac{s+1}{(s-1)(s+2)}}$$

$$s(s+1)X(s) - 2X(s) = 1$$

$$X(s)(s^2+s-2) = 1 \Rightarrow \boxed{X(s) = \frac{1}{(s-1)(s+2)}}$$

$$\frac{s+1}{(s-1)(s+2)} = \frac{A}{s-1} + \frac{B}{s+2}$$

$$\left.\frac{s+1}{s+2}\right|_{s=1} = A \qquad \left.\frac{s+1}{s-1}\right|_{s=-2} = B$$

$$A = 2/3 \qquad B = 1/3$$

$$\frac{1}{(s-1)(s+2)} = \frac{(s+2)-(s-1)}{(s-1)(s+2)}\frac{1}{3} = \frac{1/3}{s-1} - \frac{1/3}{s+2}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{s+1}{(s-1)(s+2)}\right\} = \frac{2}{3}e^{t} + \frac{1}{3}e^{-2t}$$

$$x(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s-1)(s+2)}\right\} = \frac{1}{3}e^{t} - \frac{1}{3}e^{-2t}$$

$$\boxed{\begin{aligned} x(t) &= \tfrac{1}{3}e^{t} - \tfrac{1}{3}e^{-2t} \\ y(t) &= \tfrac{2}{3}e^{t} + \tfrac{1}{3}e^{-2t} \end{aligned}}$$

2 $\begin{cases} \dfrac{dx}{dt} = x - 2y \\ \dfrac{dy}{dt} = 5x - y \end{cases}$

$x(0) = -1 \, ; \, y(0) = 2$

$$\begin{cases} sX(s) + 1 = X(s) - 2Y(s) \\ sY(s) - 2 = 5X(s) - Y(s) \end{cases}$$

$$\begin{cases} (s-1)X(s) + 2Y(s) = -1 \\ -5\,X(s) + (s+1)\,Y(s) = 2 \end{cases} \quad \begin{vmatrix} (\cdot 5) \\ (\cdot (s-1)) \end{vmatrix} \quad \begin{vmatrix} +(s+1) \\ (-2) \end{vmatrix}$$

$$10Y(s) + (s^2-1)Y(s) = -5 + 2(s-1)$$

$$(s^2+9)\,Y(s) = 2s - 7$$

$$\boxed{Y(s) = \frac{2s-7}{s^2+9}}$$

$$(s^2-1)X(s) + 10X(s) = -s - 1 - 4$$

$$(s^2+9)X(s) = -s - 5$$

$$\boxed{X(s) = -\frac{s+5}{s^2+9}}$$

$$\boxed{\begin{aligned} x(t) &= -\cos(3t) - \tfrac{5}{3}\sin(3t) \\ y(t) &= 2\cos(3t) - \tfrac{7}{3}\sin(3t) \end{aligned}}$$

3 $\begin{cases} 2\dfrac{dx}{dt} + \dfrac{dy}{dt} - 2x = 1 \\ \dfrac{dx}{dt} + \dfrac{dy}{dt} - 3x - 3y = 2 \end{cases}$

$x(0) = 0 \, ; \, y(0) = 0$

$$\begin{cases} 2sX(s) + sY(s) - 2X(s) = \frac{1}{s} \\ sX(s) + sY(s) - 3X(s) - 3Y(s) = \frac{2}{s} \end{cases}$$

$$\begin{cases} (2s-2)X(s) + sY(s) = \frac{1}{s} \\ (s-3)X(s) + (s-3)Y(s) = \frac{2}{s} \end{cases}$$

$$Y(s) = \frac{2}{s(s-3)} - X(s)$$

$$(2s-2)X(s) + \frac{2}{s-3} - sX(s) = \frac{1}{s}$$

$$(s-2)X(s) = \frac{1}{s} - \frac{2}{s-3}$$

$$X(s) = \frac{1}{s(s-2)} - \frac{2}{(s-2)(s-3)}$$

$$= \boxed{\frac{5}{2}\frac{1}{s-2} - \frac{2}{s-3} - \frac{1}{2}\frac{1}{s}}$$

$$\Rightarrow Y(s) = \frac{2}{s(s-3)} - X(s)$$

$$= \boxed{\frac{8}{3}\frac{1}{s-3} - \frac{5}{2}\frac{1}{s-2} - \frac{1}{6}\frac{1}{s}}$$

$$\boxed{\begin{aligned} x(t) &= \tfrac{5}{2}e^{2t} - 2e^{3t} - \tfrac{1}{2} \\ y(t) &= \tfrac{8}{3}e^{3t} - \tfrac{5}{2}e^{2t} - \tfrac{1}{6} \end{aligned}}$$

$$\text{4}\quad \begin{cases} \dfrac{d^2x}{dt^2} + x - y = 0 \\ \dfrac{d^2y}{dt^2} + y - x = 0 \end{cases}$$

$x(0)=0 \qquad y(0)=0$

$x'(0)=-2 \qquad y'(0)=1$

$$\begin{cases} s^2X(s) + 2 + X(s) - Y(s) = 0 \\ s^2Y(s) - 1 + Y(s) - X(s) = 0 \end{cases}$$

$$\begin{cases} (s^2+1)X(s) - Y(s) = -2 & \times 1 & \times (s^2+1) \\ -X(s) + (s^2+1)Y(s) = +1 & \times (s^2+1) & \times 1 \end{cases}$$

$$-Y(s) + (s^2+1)^2 Y(s) = -2 + s^2 + 1$$

$$((s^2+1)^2 - 1)Y(s) = s^2 - 1$$

$$Y(s) = \frac{s^2-1}{s^2(s^2+2)} = \frac{1}{s^2+2} - \frac{1}{s^2(s^2+2)}$$

$$= \frac{1}{s^2+2} - \frac{1}{2}\,\frac{(s^2+2)-s^2}{s^2(s^2+2)}$$

$$= \frac{1}{s^2+2} - \frac{1/2}{s^2} + \frac{1/2}{s^2+2}$$

$$= \frac{3/2}{s^2+2} - \frac{1/2}{s^2}$$

$$(s^2+1)^2 X(s) - X(s) = -2(s^2+1) + 1$$

$$s^2(s^2+2)X(s) = -2s^2 - 1$$

$$X(s) = -\frac{2s^2+1}{s^2(s^2+2)}$$

$$= -\frac{2}{s^2+2} - \frac{1}{s^2(s^2+2)}$$

$$= -\frac{2}{s^2+2} - \frac{1/2}{s^2} + \frac{1/2}{s^2+2}$$

$$= \frac{-3/2}{s^2+2} - \frac{1/2}{s^2}$$

$$\Rightarrow \boxed{y(t) = \frac{3}{2\sqrt{2}}\sin(\sqrt{2}t) - \frac{1}{2}t}$$

$$\Rightarrow \boxed{x(t) = -\frac{3}{2\sqrt{2}}\sin(\sqrt{2}t) - \frac{1}{2}t}$$

5. $$\begin{cases} \dfrac{d^2x}{dt^2} + \dfrac{d^2y}{dt^2} = t^2 \\ \dfrac{d^2x}{dt^2} - \dfrac{d^2y}{dt^2} = 4t \end{cases}$$

$x(0)=8;\quad y(0)=0$

$x'(0)=0;\quad y'(0)=0.$

$$\begin{cases} s^2X(s) - 8s + s^2Y(s) = \dfrac{2}{s^3} \\ s^2X(s) - 8s - s^2Y(s) = \dfrac{4}{s^2} \end{cases}$$

$$\begin{cases} s^2X(s) + s^2Y(s) = \dfrac{2}{s^3} + 8s \\ s^2X(s) - s^2Y(s) = \dfrac{4}{s^2} + 8s \end{cases}$$

$$\begin{cases} X(s) + Y(s) = \dfrac{2}{s^5} + \dfrac{8}{s} \\ X(s) - Y(s) = \dfrac{4}{s^4} + \dfrac{8}{s} \end{cases}$$

$$2Y(s) = \frac{2}{s^5} - \frac{4}{s^4}$$

$$Y(s) = \frac{1}{s^5} - \frac{2}{s^4}$$

$$\boxed{y(t) = \frac{1}{24}t^4 - \frac{1}{3}t^3}$$

$$2X(s) = \frac{2}{s^5} + \frac{4}{s^4} + \frac{16}{s}$$

$$X(s) = \frac{1}{s^5} + \frac{2}{s^4} + \frac{8}{s}$$

$$\Rightarrow \boxed{x(t) = \frac{1}{24}t^4 + \frac{1}{3}t^3 + 8}$$

6. $$\begin{cases} \dfrac{d^2x}{dt^2} + 3\dfrac{dy}{dt} + 3y = 0 \\ \dfrac{d^2x}{dt^2} + 3y = te^{-t} \end{cases}$$

$x(0)=0;\quad y(0)=0$

$x'(0)=2;$

$$\begin{cases} s^2X(s) - 2 + 3sY(s) + 3Y(s) = 0 \\ s^2X(s) - 2 + 3Y(s) = \dfrac{1}{(s+1)^2} \end{cases}$$

$$\begin{cases} s^2X(s) + 3(s+1)Y(s) = 2 \\ s^2X(s) + 3Y(s) = 2 + \dfrac{1}{(s+1)^2} \end{cases}$$

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$$3sY(s) = -\frac{1}{(s+1)^2} \Rightarrow \boxed{Y(s) = \frac{-1}{3s(s+1)^2}}$$

$$X(s) = \frac{2}{s^2} + \frac{\cancel{3}(s+1)}{s^2} \cdot \frac{1}{\cancel{3}s(s+1)^{\cancel{2}}}$$

$$= \boxed{\frac{2}{s^2} + \frac{1}{s^3(s+1)}}$$

$$\frac{1}{s(s+1)^2} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{(s+1)^2}$$

$A=1$ $C=-1$ $B=-1$

$$\frac{B}{s+1} = \frac{1}{s(s+1)^2} - \frac{1}{s} + \frac{1}{(s+1)^2}$$

$$= \frac{1-(s+1)^2+s}{s(s+1)^2} = \frac{\cancel{1}-s^2-2s-\cancel{1}+s}{s(s+1)^2} = -\frac{s+1}{(s+1)^2} = \frac{-1}{s+1}$$

$$y(t) = -\frac{1}{3}\left(1-e^{-t}-\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\}\right)$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2}\Big|_{s\to s+1}\right\} = te^{-t}$$

$$\boxed{y(t) = -\frac{1}{3}+\frac{1}{3}e^{-t}+\frac{1}{3}te^{-t}}$$

$$\frac{1}{s^3(s+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{s+1}$$

$D=-1$ $C=1$

$$\frac{A}{s}+\frac{B}{s^2} = \frac{1}{s^3(s+1)} - \frac{1}{s^3} + \frac{1}{s+1} = \frac{\cancel{1}-s-\cancel{1}+s^3}{s^3(s+1)} = \frac{s(s-1)\cancel{(s+1)}}{s^3\cancel{(s+1)}}$$

$$= \frac{s-1}{s^2} = \frac{1}{s} - \frac{1}{s^2}$$

$A=1$ $B=-1$

$$\Rightarrow x(t) = 2t+1-t+\frac{1}{2}t^2-e^{-t}$$

$$\boxed{x(t) = 1+t+\frac{1}{2}t^2-e^{-t}}$$

Another way to solve #6 :

$$\begin{cases} \dfrac{d^2x}{dt^2} + 3\dfrac{dy}{dt} + 3y = 0 & (1) \\ \dfrac{d^2x}{dt^2} + 3y = te^{-t} & (2) \end{cases}$$

$x(0)=0$; $y(0)=0$
$x'(0)=2$;

$$(1)-(2) \Rightarrow 3\frac{dy}{dt} = -te^{-t} \Rightarrow \frac{dy}{dt} = -\frac{1}{3}te^{-t}$$

$$\Rightarrow y(t) = -\frac{1}{3}\int te^{-t}dt$$
$$= \frac{1}{3}\int t(e^{-t})'dt$$
$$= \frac{1}{3}\left(te^{-t} - \int e^{-t}dt\right)$$
$$= \frac{1}{3}(te^{-t} + e^{-t} + c)$$

$$y(0)=0 \Rightarrow 0 = \frac{1}{3}(1+c) \Rightarrow c=-1 \Rightarrow \boxed{y(t) = \frac{1}{3}te^{-t} + \frac{1}{3}e^{-t} - \frac{1}{3}}$$

$$\text{Sub in (2)} \Rightarrow \frac{d^2x}{dt^2} = te^{-t} - 3y \Rightarrow \frac{d^2x}{dt^2} = -e^{-t} + 1$$
$$\Rightarrow \frac{dx}{dt} = e^{-t} + t + C_1$$
$$\Rightarrow x(t) = -e^{-t} + \frac{t^2}{2} + C_1t + C_2$$

$x'(0)=2 \Rightarrow 2 = 1 + C_1 \Rightarrow C_1 = 1$
$x(0)=0 \Rightarrow 0 = -1 + C_2 \Rightarrow C_2 = 1$

$$\Rightarrow \boxed{x(t) = -e^{-t} + \frac{t^2}{2} + t + 1}$$

7 $\mathcal{L}\{t^{3/2}\}$

$$\mathcal{L}\{t^{3/2}\} = \frac{\Gamma(5/2)}{s^{5/2}}$$

$$\Gamma(5/2) = \Gamma(3/2+1) = \frac{3}{2}\Gamma(3/2) = \frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi} = \frac{3}{4}\sqrt{\pi} \Rightarrow \boxed{\mathcal{L}\{t^{3/2}\} = \frac{3\sqrt{\pi}}{4s^{5/2}}}$$

8 $\mathcal{L}\{t^{5/2}\}$

$$\mathcal{L}\{t^{5/2}\} = \frac{\Gamma(7/2)}{s^{7/2}}$$

$$\Gamma(7/2) = \Gamma(5/2+1) = \frac{5}{2}\Gamma(5/2) = \frac{5}{2}\cdot\frac{3}{4}\sqrt{\pi} = \frac{15}{8}\sqrt{\pi} \Rightarrow \boxed{\mathcal{L}\{t^{5/2}\} = \frac{15\sqrt{\pi}}{8s^{7/2}}}$$

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9 $\mathcal{L}\{t^{\frac{2n-1}{2}}\} = \dfrac{(2n-1)!!\,\sqrt{\pi}}{2^n s^{\frac{2n+1}{2}}}$ (*)

$\mathcal{L}\{t^{1/2}\} = \dfrac{\sqrt{\pi}}{2s^{3/2}}$ (*) for $n=1$ ✓

$\mathcal{L}\{t^{3/2}\} = \dfrac{3\sqrt{\pi}}{2^2 s^{5/2}}$ (*) for $n=2$ ✓

$\mathcal{L}\{t^{5/2}\} = \dfrac{5\cdot 3\sqrt{\pi}}{2^3 s^{7/2}}$ (*) for $n=3$ ✓

General Induction: Suppose (*) is true for some positive integer n, i.e.

$$\mathcal{L}\{t^{\frac{2n-1}{2}}\} = \frac{(2n-1)!!\,\sqrt{\pi}}{2^n s^{\frac{2n+1}{2}}}$$

Show the statement is true for $(n+1)$, i.e. show that:

$$\boxed{\mathcal{L}\{t^{\frac{2n+1}{2}}\} = \frac{(2n+1)!!\,\sqrt{\pi}}{2^{n+1} s^{\frac{2n+3}{2}}}} \quad (**)$$

$$\mathcal{L}\left\{t^{\frac{2n+1}{2}}\right\} = \mathcal{L}\left\{t \cdot t^{\frac{2n-1}{2}}\right\} = (-1)\,\frac{d}{ds}\,\mathcal{L}\left\{t^{\frac{2n-1}{2}}\right\}$$

$$= -\frac{d}{ds}\;\frac{(2n-1)!!\sqrt{\pi}}{2^n s^{\frac{2n+1}{2}}}$$

$$= -\frac{(2n-1)!!\sqrt{\pi}}{2^n}\left(\frac{d}{ds}\, s^{-\frac{2n+1}{2}}\right)$$

$$= -\frac{(2n-1)!!\sqrt{\pi}}{2^n}\left(-\frac{2n+1}{2}\right) s^{-\frac{2n+1}{2}-1}$$

$$= \frac{(2n+1)!!\sqrt{\pi}}{2^{n+1}}\, s^{-\frac{2n+3}{2}}$$