Math 2552 - Differential Equations **Homework 17**
Sections F1 – F4; L1 – L4
Georgia Institute of Technology, Fall 2015

A. Solving Systems of Differential Equations with the Laplace Transform

Use the Laplace transform method to solve the following systems of differential equations:

1. $\begin{cases} \frac{dx}{dt} = -x + y \\ \frac{dy}{dt} = 2x \end{cases}$
$x(0) = 0;\ y(0) = 1.$

2. $\begin{cases} \frac{dx}{dt} = x - 2y \\ \frac{dy}{dt} = 5x - y \end{cases}$
$x(0) = -1;\ y(0) = 2.$

3. $\begin{cases} 2\frac{dx}{dt} + \frac{dy}{dt} - 2x = 1 \\ \frac{dx}{dt} + \frac{dy}{dt} - 3x - 3y = 2 \end{cases}$
$x(0) = 0;\ y(0) = 0.$

4. $\begin{cases} \frac{d^2x}{dt^2} + x - y = 0 \\ \frac{d^2y}{dt^2} + y - x = 0 \end{cases}$
$x(0) = 0;\ x'(0) = -2;$
$y(0) = 0; y'(0) = 1.$

5. $\begin{cases} \frac{d^2x}{dt^2} + \frac{d^2y}{dt^2} = t^2 \\ \frac{d^2x}{dt^2} - \frac{d^2y}{dt^2} = 4t \end{cases}$
$x(0) = 8;\ x'(0) = 0;$
$y(0) = 0;\ y'(0) = 0.$

6. $\begin{cases} \frac{d^2x}{dt^2} + 3\frac{dy}{dt} + 3y = 0 \\ \frac{d^2x}{dt^2} + 3y = te^{-t} \end{cases}$
$x(0) = 0;\ x'(0) = 2;$
$y(0) = 0.$

B. Laplace Transforms and The Gamma Function

Recall that we used $\Gamma(1/2) = \sqrt{\pi}$ to compute the Laplace transform $\mathscr{L}\{t^{1/2}\}$ – we essentially just had to figure out $\Gamma(3/2)$. Use the same technique to compute:

7. $\mathscr{L}\{t^{3/2}\}$

8. $\mathscr{L}\{t^{5/2}\}$

At this point you have proved several instances of the formula:

$$\mathscr{L}\left\{t^{\frac{2n-1}{2}}\right\} = \frac{(2n-1)!!\sqrt{\pi}}{2^n s^{\frac{2n+1}{2}}}, \qquad (\star)$$

for all positive integers n, where $(2n-1)!!$ is the double factorial:

$$(2n-1)!! = 1 \cdot 3 \cdot 5 \cdot 7 \ldots \cdot (2n-1).$$

9. Use induction to prove that $(\star)$ holds for all positive integers n. That is, verify that $(\star)$ is true for some initial value of n (we proved the case $n = 1$ in class, and above you have the cases $n = 2$ and $n = 3$, in exercises 7 and 8). Then, assume that $(\star)$ is true for some integer n and show that this implies $(\star)$ is also true for $n + 1$.

Laplace transforms of some basic functions

$\mathscr{L}\{1\} = \frac{1}{s};\;\; s > 0$

$\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}};\;\; s > 0$

$\mathscr{L}\{e^{kt}\} = \frac{1}{s-k};\;\; s > k$

$\mathscr{L}\{\sin(kt)\} = \frac{k}{s^2+k^2};\;\; s > 0$

$\mathscr{L}\{\cos(kt)\} = \frac{s}{s^2+k^2};\;\; s > 0$

$\mathscr{L}\{\sinh(kt)\} = \frac{k}{s^2-k^2};\;\; s > |k|$

$\mathscr{L}\{\cosh(kt)\} = \frac{s}{s^2-k^2};\;\; s > |k|$

$\mathscr{L}\{u_a(t)\} = \frac{e^{-as}}{s};\; s > 0$

$\mathscr{L}\{\delta(t-t_0)\} = e^{-st_0}$

$\mathscr{L}\{\delta(t)\} = 1$

Inverse Laplace transforms of some basic functions

$\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} = 1$

$\mathscr{L}^{-1}\left\{\frac{1}{s^n}\right\} = \frac{1}{(n-1)!}t^{n-1}$

$\mathscr{L}^{-1}\left\{\frac{1}{s-k}\right\} = e^{kt}$

$\mathscr{L}^{-1}\left\{\frac{1}{s^2+k^2}\right\} = \frac{1}{k}\sin(kt)$

$\mathscr{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\} = \cos(kt)$

$\mathscr{L}^{-1}\left\{\frac{1}{s^2-k^2}\right\} = \frac{1}{k}\sinh(kt)$

$\mathscr{L}^{-1}\left\{\frac{s}{s^2-k^2}\right\} = \cosh(kt)$

$\mathscr{L}^{-1}\left\{\frac{e^{-as}}{s}\right\} = u_a(t)$

$\mathscr{L}^{-1}\left\{e^{-st_0}\right\} = \delta(t-t_0)$

$\mathscr{L}^{-1}\{1\} = \delta(t)$

Properties of the Laplace and Inverse Laplace transform

Translation Theorem I:

$$\mathscr{L}\left\{e^{kt}f(t)\right\} = F(s-k) = \mathscr{L}\{f(t)\}\big|_{s\to s-k}$$

$$\mathscr{L}^{-1}\{F(s-k)\} = \mathscr{L}^{-1}\left\{F(s)|_{s\to s-k}\right\} = e^{kt}\mathscr{L}^{-1}\{F(s)\} = e^{kt}f(t)$$

Translation Theorem II:

$$\mathscr{L}\left\{f(t-a)u_a(t)\right\} = e^{-as}F(s) = e^{-as}\mathscr{L}\{f(t)\}$$

$$\mathscr{L}^{-1}\{e^{-as}F(s)\} = f(t-a)u_a(t) = \mathscr{L}^{-1}\{F(s)\}\big|_{t\to t-a}u_a(t)$$

Derivatives of Laplace Transforms:

$$\mathscr{L}\left\{t^n f(t)\right\} = (-1)^n\frac{d^n}{ds^n}F(s)$$

$$\mathscr{L}^{-1}\left\{F^{(n)}(s)\right\} = (-1)^n t^n f(t)$$

Laplace Transform of Periodic Functions:
If $f(t)$ is piecewise continuous on $[0,\infty)$, of exponential order, and periodic with period T:

$$\mathscr{L}\{f(t)\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)\,dt$$

Laplace Transforms of Derivatives:

$$\begin{aligned}
\mathscr{L}\{y'\} &= sY(s) - y(0)\\
\mathscr{L}\{y''\} &= s^2Y(s) - sy(0) - y'(0)\\
&\;\;\vdots\\
\mathscr{L}\{y^{(n)}(t)\} &= s^nY(s) - s^{n-1}y(0) - s^{n-2}y'(0) - \ldots - sy^{(n-2)}(0) - y^{(n-1)}(0)
\end{aligned}$$

Convolution Theorem:

$$\mathscr{L}\{f(t)\star g(t)\} = F(s)G(s) = \mathscr{L}\{f(t)\}\mathscr{L}\{g(t)\}$$

$$\mathscr{L}^{-1}\{F(s)G(s)\} = f(t)\star g(t)$$

Dirac Delta Function:

$$\int_0^\infty f(t)\delta(t-t_0)\,dt = f(t_0)$$

$$(f\star\delta)(t) = f(t)$$

Solving Systems of ODEs with the Laplace Transform

$$\begin{cases} y_1'' + 10y_1 - 4y_2 = 0 \\ y_2'' - 4y_1 + 4y_2 = 0 \end{cases}$$

$y_1(0) = 0$; $y_1'(0) = 1$

$y_2(0) = 0$; $y_2'(0) = -1$.

Take Laplace of both sides in both equations, solve for $Y_1(s)$ and $Y_2(s)$, and then take inverse Laplace to find $y_1(t)$ and $y_2(t)$.

$$\begin{cases} s^2 Y_1(s) - 1 + 10Y_1(s) - 4Y_2(s) = 0 \\ s^2 Y_2(s) + 1 - 4Y_1(s) + 4Y_2(s) = 0 \end{cases}$$

$$\begin{cases} (s^2+10)\,Y_1(s) - 4\,Y_2(s) = 1 \\ -4\,Y_1(s) + (s^2+4)\,Y_2(s) = -1 \end{cases} \quad \begin{vmatrix} \times (s^2+4) \\ \times 4 \end{vmatrix} \quad \begin{vmatrix} \times 4 \\ \times (s^2+10) \end{vmatrix}$$

$$(s^2+10)(s^2+4)Y_1(s) - 16Y_1(s) = s^2 + 4 - 4$$

$$(s^4 + 14s^2 + 24)\,Y_1(s) = s^2$$

$$\boxed{Y_1(s) = \frac{s^2}{(s^2+2)(s^2+12)}}$$

$$= \frac{-1/5}{s^2+2} + \frac{6/5}{s^2+12}$$

$$-16Y_2(s) + (s^2+4)(s^2+10)Y_2(s) = 4 - s^2 - 10$$

$$(s^4 + 14s^2 + 24)\,Y_2(s) = -s^2 - 6$$

$$\boxed{Y_2(s) = -\frac{s^2+6}{(s^2+2)(s^2+12)}}$$

$$= \frac{-2/5}{s^2+2} - \frac{3/5}{s^2+12}$$

$$\Rightarrow \boxed{y_1(t) = -\frac{1}{5\sqrt{2}}\sin(\sqrt{2}t) + \frac{\sqrt{3}}{5}\sin(2\sqrt{3}t)} \;\&\; \boxed{y_2(t) = -\frac{2}{5\sqrt{2}}\sin(\sqrt{2}t) - \frac{\sqrt{3}}{10}\sin(2\sqrt{3}t)}$$

The Gamma Function

- The Gamma function is defined for all $x>0$ by the integral:

$$\boxed{\Gamma(x) := \int_0^\infty e^{-t} t^{x-1}\,dt}$$

- This function serves as a continuous analogue to the factorial:

$$\Gamma(x+1) = \int_0^\infty e^{-t} t^x\,dt = \int_0^\infty (-e^{-t})' t^x\,dt$$

$x>0 \Rightarrow x+1>1$

$$= \underbrace{-e^{-t} t^x \Big|_{t=0}^{\infty}}_{0} + \int_0^\infty e^{-t} \cdot x t^{x-1}\,dt$$

$$= x \underbrace{\int_0^\infty e^{-t} t^{x-1}\,dt}_{\Gamma(x)} \Rightarrow \boxed{\Gamma(x+1) = x\Gamma(x)} \qquad (*)$$

- Compute $\Gamma(1)$:

$$\Gamma(1) = \int_0^\infty e^{-t}\,dt = -e^{-t}\Big|_0^\infty = 1 \Rightarrow \Gamma(1)=1$$

Using $(*)$:

$$\Gamma(2) = 1\cdot\Gamma(1) = 1 = 1!$$
$$\Gamma(3) = 2\cdot\Gamma(2) = 2\cdot 1 = 2!$$
$$\Gamma(4) = 3\cdot\Gamma(3) = 3\cdot 2\cdot 1 = 3!$$
$$\Gamma(5) = 4\cdot\Gamma(4) = 4\cdot 3\cdot 2\cdot 1 = 4!$$
$$\vdots$$

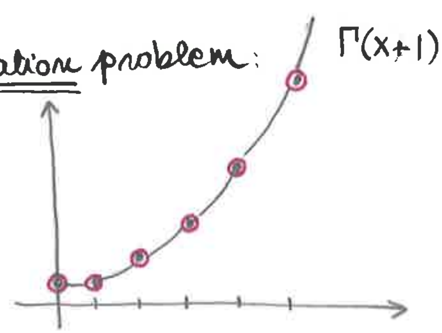
$$\boxed{\Gamma(n) = (n-1)!} \quad \text{for all positive integers } n$$

- Note that for $n=1$: $\Gamma(1) = 0!$ and we computed $\Gamma(1)=1$.

  $\Rightarrow$ reason why $0! = 1$.

- $\Gamma(x)$ can be thought of as a solution to the interpolation problem:

  Suppose we are given the set of points in the plane: $(n, n!)$ for all integers $n = 0,1,2,\dots$ Is there a continuous function that "joins" all of these points? Yes: $f(x) = \Gamma(x+1)$.



The Gamma Function and the Laplace Transform:

Let us try to compute the Laplace transform of a general power function, i.e. $f(t)=t^p,\ p\geq 0$:

$$\mathcal{L}\{t^p\} = \int_0^\infty e^{-st}t^p\,dt$$

Change of variable: $r = st$, $dr = s\,dt$

$$= \int_0^\infty e^{-r}\left(\frac{r}{s}\right)^p \frac{1}{s}\,dr$$

$$= \frac{1}{s^{p+1}}\underbrace{\int_0^\infty e^{-r}r^p\,dr}_{\Gamma(p+1)} \Rightarrow \boxed{\mathcal{L}\{t^p\} = \frac{\Gamma(p+1)}{s^{p+1}}} \quad (\forall)\,p>-1$$

- If p is a positive integer $p=n$: $\mathcal{L}\{t^n\} = \dfrac{\Gamma(n+1)}{s^{n+1}} = \dfrac{n!}{s^{n+1}}$ (the formula we already know).

$$\boxed{\Gamma(1/2) = \sqrt{\pi}}$$

$$\Gamma(1/2) = \int_0^\infty e^{-r}r^{-1/2}\,dr$$

$$= \int_0^\infty e^{-r}\frac{1}{\sqrt{r}}\,dr$$

Change of variable: $x=\sqrt{r} \Rightarrow dx = \dfrac{1}{2\sqrt{r}}\,dr$

$$= \int_0^\infty e^{-x^2}2\,dx$$

$$= 2\int_0^\infty e^{-x^2}dx$$

Gaussian Integral: $\boxed{\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}}$

$$= 2\cdot\frac{\sqrt{\pi}}{2} = \boxed{\sqrt{\pi}}$$

$$\Rightarrow \boxed{\mathcal{L}\{t^{-1/2}\} = \frac{\sqrt{\pi}}{s^{1/2}}}$$

$$\mathcal{L}\{t^{-1/2}\} = \frac{\Gamma(1/2)}{s^{1/2}} = \frac{\sqrt{\pi}}{s^{1/2}}$$

$$\Rightarrow \boxed{\mathcal{L}\{t^{1/2}\} = \frac{\sqrt{\pi}}{2s^{3/2}}}$$

$$\frac{\sqrt{\pi}}{s^{1/2}} = \mathcal{L}\{t^{-1/2}\} = \mathcal{L}\left\{\frac{1}{\sqrt{t}}\right\} = 2\mathcal{L}\{y'(t)\} \quad \text{where } y(t)=\sqrt{t}$$

$$= 2\left(sY(s) - y(0)\right)$$

$$= 2s\,\mathcal{L}\{\sqrt{t}\}$$

$$\Rightarrow \mathcal{L}\{\sqrt{t}\} = \frac{1}{2s}\,\frac{\sqrt{\pi}}{\sqrt{s}} = \frac{\sqrt{\pi}}{2s^{3/2}}$$

Another way to show $\mathcal{L}\{\sqrt{t}\} = \dfrac{\sqrt{\pi}}{2s^{3/2}}$:

$$\mathcal{L}\{t^{1/2}\} = \frac{\Gamma(3/2)}{s^{3/2}}$$

How to find $\Gamma(3/2)$?

$$\Gamma(3/2) = \Gamma(1+1/2) = \frac{1}{2}\Gamma(1/2) = \frac{1}{2}\sqrt{\pi} \Rightarrow \mathcal{L}\{t^{1/2}\} = \frac{\sqrt{\pi}}{2s^{3/2}}$$