1 $$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+k^2)^2}\right\} = \mathcal{L}^{-1}\{F(s)G(s)\} \quad \text{where } F(s)=G(s)=\frac{1}{s^2+k^2}$$

$$= (f*g)(t) \quad \text{where } f(t)=g(t)=\frac{1}{k}\sin(kt)$$

$$\Rightarrow (f*g)(t) = \frac{1}{k^2}\int_0^t \sin(kt-k\tau)\sin(k\tau)\,d\tau$$

$$= \frac{1}{k^2}\int_0^t \frac{1}{2}\Big(\cos(kt-2k\tau)-\cos(kt)\Big)\,d\tau$$

$$= \frac{1}{2k^2}\left(-\frac{1}{2k}\sin(kt-2k\tau)\Big|_0^t - t\cos(kt)\right)$$

$$= \frac{1}{2k^2}\left(-\frac{1}{2k}\sin(-kt)+\frac{1}{2k}\sin(kt)-t\cos(kt)\right)$$

$$= \frac{1}{2k^2}\left(\frac{1}{k}\sin(kt)-t\cos(kt)\right) = \boxed{\frac{\sin(kt)-kt\cos(kt)}{2k^3}}$$

2 $$\mathcal{L}^{-1}\left\{\frac{G(s)}{(s-1)^2+1}\right\} = \mathcal{L}^{-1}\{F(s)G(s)\} \quad \text{where } F(s)=\frac{1}{(s-1)^2+1}$$

$$= (f*g)(t) \quad \text{where } f(t)=\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2+1}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\Big|_{s\to s-1}\right\}$$

$$= e^t\sin t$$

$$\Rightarrow \boxed{\mathcal{L}^{-1}\left\{\frac{G(s)}{(s-1)^2+1}\right\} = e^t\sin t * g(t)}$$

③ $y'' + k^2 y = g(t)$; $y(0) = \alpha$; $y'(0) = \beta$

$k \neq 0$; $\alpha, \beta \in \mathbb{R}$
$g(t)$ piecewise continuous on $[0, \infty)$, of exponential order

$$s^2 Y(s) - s y(0) - y'(0) + k^2 Y(s) = G(s)$$

$$(s^2 + k^2) Y(s) = \alpha s + \beta + G(s)$$

$$Y(s) = \frac{\alpha s}{s^2 + k^2} + \frac{\beta}{s^2 + k^2} + \frac{G(s)}{s^2 + k^2}$$

$$y(t) = \alpha \mathcal{L}^{-1}\left\{\frac{s}{s^2 + k^2}\right\} + \beta \mathcal{L}^{-1}\left\{\frac{1}{s^2 + k^2}\right\} + \mathcal{L}^{-1}\left\{\frac{G(s)}{s^2 + k^2}\right\}$$

$$= \alpha \cos(kt) + \beta \frac{1}{k} \sin(kt) + (f * g)(t)$$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2 + k^2}\right\} = \frac{1}{k} \sin(kt)$$

$$\Rightarrow \boxed{y(t) = \alpha \cos(kt) + \frac{\beta}{k} \sin(kt) + \frac{1}{k} \sin(kt) * g(t)}$$

④ Volterra integral equation: $x(t) = 3\cos t + 5 \int_0^t \sin(t - \tau) x(\tau) d\tau$

$$\Rightarrow X(s) = \frac{3s}{s^2 + 1} + 5 \mathcal{L}\{\sin t * x(t)\}$$

$$= \frac{3s}{s^2 + 1} + \frac{5}{s^2 + 1} X(s) \quad \Rightarrow \left(1 - \frac{5}{s^2 + 1}\right) X(s) = \frac{3s}{s^2 + 1}$$

$$\Rightarrow X(s) = \frac{3s}{s^2 - 4} \quad \Rightarrow x(t) = \mathcal{L}^{-1}\left\{\frac{3s}{s^2 - 4}\right\} = 3 \mathcal{L}^{-1}\left\{\frac{s}{(s-2)(s+2)}\right\}$$

$$= 3 \mathcal{L}^{-1}\left\{\frac{1}{2} \frac{1}{s-2} + \frac{1}{2} \frac{1}{s+2}\right\}$$

$$= \boxed{\frac{3}{2}\left(e^{2t} + e^{-2t}\right)}$$

⑤ $y(t) = t^3 + \int_0^t (t-\tau)\, y(\tau)\, d\tau$

$y(t) = t^3 + (f * y)(t)$, where $f(t) = t \Rightarrow F(s) = \frac{1}{s^2}$

$\Rightarrow \mathcal{L}\{y(t)\} = \mathcal{L}\{t^3\} + \mathcal{L}\{(f*y)(t)\}$

$\Rightarrow Y(s) = \frac{6}{s^4} + F(s)Y(s)$

$= \frac{6}{s^4} + \frac{1}{s^2} Y(s) \quad \Rightarrow \left(1 - \frac{1}{s^2}\right) Y(s) = \frac{6}{s^4}$

$\Rightarrow \frac{s^2-1}{s^2} Y(s) = \frac{6}{s^4} \Rightarrow \boxed{Y(s) = \frac{6}{s^2(s^2-1)}}$

$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2-1)}\right\} = \mathcal{L}^{-1}\left\{\frac{-1}{s^2} + \frac{1/2}{s-1} - \frac{1/2}{s+1}\right\} = -t + \frac{1}{2}e^t - \frac{1}{2}e^{-t}$

$\frac{1}{s^2(s^2-1)} = \frac{1}{s^2(s-1)(s+1)}$

$= \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-1} + \frac{D}{s+1}$

$\left.\frac{1}{(s-1)(s+1)}\right|_{s=0} = B \Rightarrow B = -1$

$\left.\frac{1}{s^2(s+1)}\right|_{s=1} = C \Rightarrow C = 1/2$

$\left.\frac{1}{s^2(s-1)}\right|_{s=-1} = D \Rightarrow D = -1/2$

$\Rightarrow 1 = As - (s^2-1) + \frac{1}{2}s^2(s+1) - \frac{1}{2}s^2(s-1)$

Match coefficients of s:

$0 = A$

$\Rightarrow \boxed{y(t) = -6t + 3e^t - 3e^{-t}}$

6. $y(t) = e^t + \int_0^t y(\tau)\,d\tau$

$= e^t + (f*y)(t)$, where $f(t) = 1$. $\Rightarrow F(s) = \frac{1}{s}$

$\mathcal{L}\{y(t)\} = \mathcal{L}\{e^t\} + \mathcal{L}\{(f*y)(t)\}$

$Y(s) = \frac{1}{s-1} + F(s)\,Y(s)$

$= \frac{1}{s-1} + \frac{1}{s}Y(s) \quad \Rightarrow Y(s)\left(1 - \frac{1}{s}\right) = \frac{1}{s-1}$

$\Rightarrow Y(s)\,\frac{s-1}{s} = \frac{1}{s-1}$

$\Rightarrow \boxed{Y(s) = \frac{s}{(s-1)^2}}$

$\Rightarrow y(t) = \mathcal{L}^{-1}\left\{\frac{s}{(s-1)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{s-1+1}{(s-1)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s-1} + \frac{1}{(s-1)^2}\right\}$

$= e^t + \mathcal{L}^{-1}\left\{\frac{1}{s^2}\Big|_{s\to s-1}\right\}$

$= e^t + e^t\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = e^t(1+t) \qquad \boxed{y(t) = e^t(1+t)}$

7. $y(t) = 1 + \int_0^t (\tau - t)\,y(\tau)\,d\tau$

$= 1 + (f*y)(t)$, where $f(t) = -t \qquad \Rightarrow F(s) = -\frac{1}{s^2}$

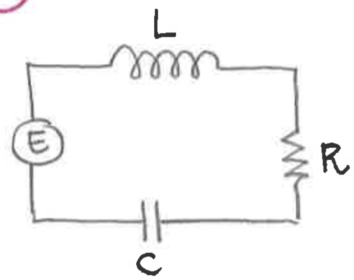
$\Rightarrow Y(s) = \frac{1}{s} - \frac{1}{s^2}Y(s) \quad \Rightarrow \quad \frac{s^2+1}{s^2}Y(s) = \frac{1}{s} \Rightarrow Y(s) = \frac{s}{s^2+1} \Rightarrow \boxed{y(t) = \cos t}$

8. $y(t) = \int_0^t (\tau - t)\,y(\tau)\,d\tau$

$= (f*y)(t)$, where $f(t) = -t \qquad \Rightarrow F(s) = -\frac{1}{s^2}$

$\Rightarrow Y(s) = -\frac{1}{s^2}Y(s) \quad \Rightarrow \quad \frac{s^2+1}{s^2}Y(s) = 0 \Rightarrow Y(s) = 0 \Rightarrow \boxed{y(t) = 0}$

9

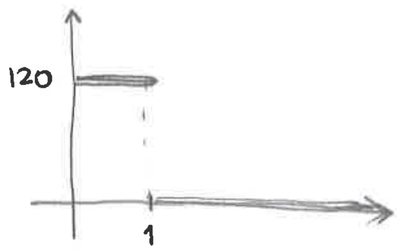


(RLC Circuit):

$$L\frac{di}{dt}+Ri(t)+\frac{1}{C}\int_0^t i(\tau)\,d\tau=E(t)$$

$L=0.1\,h$; $R=2\,\Omega$; $C=0.1\,f$; $i(0)=0$

$E(t)=120t-120t\,u_1(t)$



$$0.1\,i'+2i+10\int_0^t i(\tau)\,d\tau=120t\,(1-u_1(t))$$

$$0.1\big(s\,I(s)-i(0)\big)+2I(s)+10\,\mathcal{L}\{f*i\}$$

where $f(t)=1 \Rightarrow F(s)=\frac{1}{s}$

$$0.1\,s\,I(s)+2I(s)+10\,\frac{1}{s}\,I(s)$$

$$\mathcal{L}\{120t(1-u_1(t))\}=120\Big(\mathcal{L}\{t\}-\mathcal{L}\{(t-1+1)u_1(t)\}\Big)$$
$$=120\Big(\frac{1}{s^2}-e^{-s}\mathcal{L}\{t+1\}\Big)$$
$$=120\Big(\frac{1}{s^2}-e^{-s}\Big(\frac{1}{s^2}+\frac{1}{s}\Big)\Big)$$

$$\Rightarrow\Big(0.1s+2+\frac{10}{s}\Big)I(s)=120\,\frac{1-e^{-s}-se^{-s}}{s^2}\Rightarrow I(s)=120\,\frac{1-e^{-s}-se^{-s}}{(0.1s^2+2s+10)s}$$
$$=1200\,\frac{1-e^{-s}-se^{-s}}{(s^2+20s+100)s}$$

$$\Rightarrow i(t)=1200\,\mathcal{L}^{-1}\left\{\frac{1-e^{-s}-se^{-s}}{s(s+10)^2}\right\}$$
$$=1200\,\mathcal{L}^{-1}\left\{\frac{1}{s(s+10)^2}-\frac{1}{s(s+10)^2}e^{-s}-\frac{1}{(s+10)^2}e^{-s}\right\}$$

$$\frac{1}{s(s+10)^2} = \frac{A}{s} + \frac{B}{s+10} + \frac{C}{(s+10)^2}$$

$$\left.\frac{1}{(s+10)^2}\right|_{s=0} = A \Rightarrow A = \frac{1}{100}$$

$$\left.\frac{1}{s}\right|_{s=-10} = C \Rightarrow C = -\frac{1}{10}$$

$$1 = \frac{1}{100}(s+10)^2 + Bs(s+10) - \frac{1}{10}s$$

Match coefficients of s: $0 = \frac{1}{100}\cdot 20 + 10B - \frac{1}{10} = \frac{1}{10} + 10B \Rightarrow B = -\frac{1}{100}$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{s(s+10)^2}\right\} = \frac{1}{100} - \frac{1}{100}e^{-10t} - \frac{1}{10}e^{-10t}t$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{s(s+10)^2}e^{-s}\right\} = \left.\mathcal{L}^{-1}\left\{\frac{1}{s(s+10)^2}\right\}\right|_{t\to t-1} u_1(t)$$

$$= \left(\frac{1}{100} - \frac{1}{100}e^{-10(t-1)} - \frac{1}{10}e^{-10(t-1)}(t-1)\right)u_1(t)$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+10)^2}e^{-s}\right\} = \left.\mathcal{L}^{-1}\left\{\frac{1}{(s+10)^2}\right\}\right|_{t\to t-1} u_1(t)$$

$$= \left.\left(e^{-10t}t\right)\right|_{t\to t-1} u_1(t) = e^{-10(t-1)}(t-1)\,u_1(t)$$

$$\Rightarrow i(t) = 1200\left(\frac{1}{100} - \frac{1}{100}e^{-10t} - \frac{1}{10}e^{-10t}t \quad - \left(\frac{1}{100} - \frac{1}{100}e^{-10(t-1)} - \frac{1}{10}e^{-10(t-1)}(t-1) + e^{-10(t-1)}(t-1)\right)u_1(t)\right)$$

$$\boxed{i(t) = 12 - 12e^{-10t} - 120te^{-10t} - \left(12 - 12e^{-10(t-1)} + 1080e^{-10(t-1)}(t-1)\right)u_1(t)}$$

(10) $y' - 3y = \delta(t-2)\ ;\ y(0)=0$

$$sY(s) - 3Y(s) = e^{-2s} \Rightarrow Y(s) = \frac{e^{-2s}}{s-3}$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s-3}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\}\Big|_{t \to t-2} u_2(t)$$

$$\boxed{y(t) = e^{3t-6}\, u_2(t)}$$

(11) $y'' + y = \delta(t-2\pi)\ ;\ y(0)=0,\ y'(0)=1$

$$s^2 Y(s) - 1 + Y(s) = e^{-2\pi s}$$

$$Y(s) = \frac{1}{s^2+1} + \frac{e^{-2\pi s}}{s^2+1} \Rightarrow y(t) = \sin t + \sin(t-2\pi)\, u_{2\pi}(t)$$

$$\boxed{y(t) = \sin t\,(1 + u_{2\pi}(t))}$$

(12) $y'' + y = \delta\left(t - \frac{\pi}{2}\right) + \delta\left(t - \frac{3\pi}{2}\right)\ ;\ y(0) = y'(0) = 0.$

$$s^2 Y(s) + Y(s) = e^{-\pi/2\, s} + e^{-3\pi/2\, s} \Rightarrow Y(s) = \frac{e^{-\pi/2\, s}}{s^2+1} + \frac{e^{-3\pi/2\, s}}{s^2+1}$$

$$\Rightarrow y(t) = \sin\left(t - \frac{\pi}{2}\right) u_{\pi/2}(t) + \sin\left(t - \frac{3\pi}{2}\right) u_{3\pi/2}(t)$$

$$\boxed{y(t) = -\cos(t)\, u_{\pi/2}(t) + \cos(t)\, u_{3\pi/2}(t)}$$

(13) $y'' + 2y' = \delta(t-1)\ ;\ y(0)=0,\ y'(0)=1.$

$$s^2 Y(s) - 1 + 2s Y(s) = e^{-s}$$

$$(s^2+2s) Y(s) = 1 + e^{-s} \Rightarrow Y(s) = \frac{1}{s^2+2s} + \frac{e^{-s}}{s^2+2s}$$

$$\boxed{y(t) = \frac{1}{2}(1 - e^{-2t}) + \frac{1}{2}(1 - e^{-2t+2})\, u_1(t)}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+2s}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s(s+2)}\right\}$$

$$= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{1}{s+2}\right\}$$

$$= \frac{1}{2}\left(1 - e^{-2t}\right)$$

(14) $y'' + 4y' + 5y = \delta(t-2\pi)\,;\ y(0)=0,\ y'(0)=0$

$$s^2Y(s) + 4sY(s) + 5Y(s) = e^{-2\pi s} \Rightarrow Y(s) = \frac{e^{-2\pi s}}{s^2+4s+5}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+4s+5}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2+1}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\Big|_{s\to s+2}\right\}$$

$$= e^{-2t}\sin t$$

$$\boxed{y(t) = e^{-2t+4\pi}\sin(t)\,u_{2\pi}(t)}$$

(15) $y'' + 4y' + 13y = \delta(t-\pi) + \delta(t-3\pi)\,;\ y(0)=1,\ y'(0)=0.$

$$s^2Y(s) - s + 4sY(s) - 4 + 13Y(s) = e^{-\pi s} + e^{-3\pi s}$$

$$(s^2+4s+13)Y(s) = s+4+e^{-\pi s}+e^{-3\pi s} \Rightarrow Y(s) = \frac{s+4}{s^2+4s+13} + \frac{e^{-\pi s}}{s^2+4s+13} + \frac{e^{-3\pi s}}{s^2+4s+13}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+4s+13}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2+9}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\Big|_{s\to s+2}\right\} = e^{-2t}\frac{1}{3}\sin(3t)$$

$$\mathcal{L}^{-1}\left\{\frac{s+4}{s^2+4s+13}\right\} = \mathcal{L}^{-1}\left\{\frac{s+2+2}{(s+2)^2+9}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\Big|_{s\to s+2}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\Big|_{s\to s+2}\right\}$$

$$= e^{-2t}\cos(3t) + \frac{2}{3}e^{-2t}\sin(3t).$$

$$\boxed{\begin{aligned} y(t) = e^{-2t}\cos(3t) &+ \frac{2}{3}e^{-2t}\sin(3t) \\ &+ e^{-2t+2\pi}\frac{1}{3}\sin(3t-3\pi)\,u_\pi(t) \\ &+ e^{-2t+6\pi}\frac{1}{3}\sin(3t-9\pi)\,u_{3\pi}(t) \end{aligned}}$$