Math 2552 - Differential Equations **Homework 16**
Sections F1 – F4; L1 – L4
Georgia Institute of Technology, Fall 2015

Laplace Transform:
Convolutions and the Dirac Delta Function

1. Use convolutions to find

$$\mathscr{L}^{-1}\left\{\frac{1}{(s^2+k^2)^2}\right\},$$

where $k \neq 0$ is a constant.

2. Find

$$\mathscr{L}^{-1}\left\{\frac{G(s)}{(s-1)^2+1}\right\},$$

where $G(s)$ is the Laplace transform of a piecewise continuous function $g(t)$ of exponential order.

3. Find the solution to the initial value problem:

$$y'' + k^2 y = g(t);\;\; y(0) = \alpha,\; y'(0) = \beta,$$

where $k \neq 0$, α and β are constants, and $g(t)$ is a piecewise continuous function of exponential order.

Solve the following equations – all examples of **Volterra integral equations**:

4. $x(t) = 3\cos t + 5\int_0^t \sin(t-\tau)x(\tau)\,d\tau.$

5. $y(t) = t^3 + \int_0^t (t-\tau)y(\tau)\,d\tau.$

6. $y(t) = e^t + \int_0^t y(\tau)\,d\tau.$

7. $y(t) = 1 + \int_0^t (\tau - t)y(\tau)\,d\tau.$

8. $y(t) = \int_0^t (\tau - t)y(\tau)\,d\tau.$

9. Kirchhoff's second law in a series RLC circuit (consisting of a resistor, an inductor, and a capacitor) is:

$$L\frac{di}{dt} + Ri(t) + \frac{1}{C}\int_0^t i(\tau)\,d\tau = E(t),$$

where $E(t)$ is the voltage. Find the current $i(t)$ in such an RLC circuit, where

$$L = 0.1h;\; R = 2\Omega;\; C = 0.1f;\;\; i(0) = 0,$$

and the voltage function is given by

$$E(t) = 120t(1 - u_1(t)).$$

10. $y' - 3y = \delta(t-2);\; y(0) = 0.$

11. $y'' + y = \delta(t-2\pi);\; y(0) = 0,\; y'(0) = 1.$

12. $y'' + y = \delta\left(t - \frac{\pi}{2}\right) + \delta\left(t - \frac{3\pi}{2}\right);\; y(0) = y'(0) = 0.$

13. $y'' + 2y' = \delta(t-1);\; y(0) = 0,\; y'(0) = 1.$

14. $y'' + 4y' + 5y = \delta(t-2\pi);\; y(0) = y'(0) = 0.$

15. $y'' + 4y' + 13y = \delta(t-\pi) + \delta(t-3\pi);\; y(0) = 1,\; y'(0) = 0.$

Laplace transforms of some basic functions

$\mathscr{L}\{1\} = \frac{1}{s};\ \ s > 0$

$\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}};\ \ s > 0$

$\mathscr{L}\{e^{kt}\} = \frac{1}{s-k};\ \ s > k$

$\mathscr{L}\{\sin(kt)\} = \frac{k}{s^2+k^2};\ \ s > 0$

$\mathscr{L}\{\cos(kt)\} = \frac{s}{s^2+k^2};\ \ s > 0$

$\mathscr{L}\{\sinh(kt)\} = \frac{k}{s^2-k^2};\ \ s > |k|$

$\mathscr{L}\{\cosh(kt)\} = \frac{s}{s^2-k^2};\ \ s > |k|$

$\mathscr{L}\{u_a(t)\} = \frac{e^{-as}}{s};\ s > 0$

$\mathscr{L}\{\delta(t-t_0)\} = e^{-st_0}$

$\mathscr{L}\{\delta(t)\} = 1$

Inverse Laplace transforms of some basic functions

$\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} = 1$

$\mathscr{L}^{-1}\left\{\frac{1}{s^n}\right\} = \frac{1}{(n-1)!}t^{n-1}$

$\mathscr{L}^{-1}\left\{\frac{1}{s-k}\right\} = e^{kt}$

$\mathscr{L}^{-1}\left\{\frac{1}{s^2+k^2}\right\} = \frac{1}{k}\sin(kt)$

$\mathscr{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\} = \cos(kt)$

$\mathscr{L}^{-1}\left\{\frac{1}{s^2-k^2}\right\} = \frac{1}{k}\sinh(kt)$

$\mathscr{L}^{-1}\left\{\frac{s}{s^2-k^2}\right\} = \cosh(kt)$

$\mathscr{L}^{-1}\left\{\frac{e^{-as}}{s}\right\} = u_a(t)$

$\mathscr{L}^{-1}\left\{e^{-st_0}\right\} = \delta(t-t_0)$

$\mathscr{L}^{-1}\left\{1\right\} = \delta(t)$

Properties of the Laplace and Inverse Laplace transform

Translation Theorem I:

$$\mathscr{L}\left\{e^{kt}f(t)\right\} = F(s-k) = \mathscr{L}\{f(t)\}\big|_{s\to s-k}$$

$$\mathscr{L}^{-1}\{F(s-k)\} = \mathscr{L}^{-1}\left\{F(s)|_{s\to s-k}\right\} = e^{kt}\mathscr{L}^{-1}\{F(s)\} = e^{kt}f(t)$$

Translation Theorem II:

$$\mathscr{L}\left\{f(t-a)u_a(t)\right\} = e^{-as}F(s) = e^{-as}\mathscr{L}\{f(t)\}$$

$$\mathscr{L}^{-1}\{e^{-as}F(s)\} = f(t-a)u_a(t) = \mathscr{L}^{-1}\{F(s)\}\big|_{t\to t-a}\,u_a(t)$$

Derivatives of Laplace Transforms:

$$\mathscr{L}\left\{t^n f(t)\right\} = (-1)^n\frac{d^n}{ds^n}F(s)$$

$$\mathscr{L}^{-1}\left\{F^{(n)}(s)\right\} = (-1)^n t^n f(t)$$

Laplace Transform of Periodic Functions:
If $f(t)$ is piecewise continuous on $[0,\infty)$, of exponential order, and periodic with period T:

$$\mathscr{L}\{f(t)\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)\,dt$$

Laplace Transforms of Derivatives:

$$\begin{aligned}
\mathscr{L}\{y'\} &= sY(s) - y(0)\\
\mathscr{L}\{y''\} &= s^2Y(s) - sy(0) - y'(0)\\
&\vdots\\
\mathscr{L}\{y^{(n)}(t)\} &= s^nY(s) - s^{n-1}y(0) - s^{n-2}y'(0) - \ldots - sy^{(n-2)}(0) - y^{(n-1)}(0)
\end{aligned}$$

Convolution Theorem:

$$\mathscr{L}\{f(t)\star g(t)\} = F(s)G(s) = \mathscr{L}\{f(t)\}\mathscr{L}\{g(t)\}$$

$$\mathscr{L}^{-1}\{F(s)G(s)\} = f(t)\star g(t)$$

Dirac Delta Function:

$$\int_0^\infty f(t)\delta(t-t_0)\,dt = f(t_0)$$

$$(f\star\delta)(t) = f(t)$$

Convolution $\boxed{f*g}$

Example: $y''+y=g(t)$; $y(0)=y'(0)=0.$

Second-order linear equation with constant coefficients, with some input function $g(t)$, piecewise continuous and of exponential order on $t\in[0,\infty)$.

Solve using Variation of Parameters:

y_c: $m^2+1=0 \Rightarrow m=\pm i \Rightarrow y_c=c_1\cos t+c_2\sin t.$

y_p: $$W=\begin{vmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{vmatrix}=1.$$

$$W_1=\begin{vmatrix}0 & \sin t\\ g(t) & \cos t\end{vmatrix}=-\sin t\, g(t) \Rightarrow u_1'=-\sin t\, g(t) \underset{(\text{FTC II})}{\Rightarrow} \boxed{u_1=-\int_0^t \sin\tau\, g(\tau)\,d\tau}$$

$$W_2=\begin{vmatrix}\cos t & 0\\ -\sin t & g(t)\end{vmatrix}=g(t)\cos t \Rightarrow u_2'=\cos t\, g(t) \Rightarrow \boxed{u_2=\int_0^t \cos\tau\, g(\tau)\,d\tau}$$

$$\Rightarrow y=c_1\cos t+c_2\sin t-\cos t\int_0^t \sin\tau\, g(\tau)\,d\tau+\sin t\int_0^t\cos\tau\, g(\tau)\,d\tau$$

$$\Rightarrow y'=-c_1\sin t+c_2\cos t+\sin t\int_0^t\sin\tau\, g(\tau)\,d\tau-\cancel{\cos t\sin t\, g(t)}$$
$$+\cos t\int_0^t\cos\tau\, g(\tau)\,d\tau+\cancel{\sin t\cos t\, g(t)}$$

$$\left.\begin{array}{l} y(0)=0\Rightarrow c_1=0\\ y'(0)=0\Rightarrow c_2=0\end{array}\right\} \Rightarrow y(t)=-\cos t\int_0^t\sin\tau\, g(\tau)\,d\tau+\sin t\int_0^t\cos\tau\, g(\tau)\,d\tau$$
$$=\int_0^t(\sin t\cos\tau-\sin\tau\cos t)\,g(\tau)\,d\tau$$
$$=\int_0^t\sin(t-\tau)\,g(\tau)\,d\tau$$

$(\sin(a-b)=\sin a\cos b-\sin b\cos a)$

$\Rightarrow$ Solution: $\boxed{y(t)=\int_0^t\sin(t-\tau)\,g(\tau)\,d\tau}$

$\hookrightarrow$ convolution integral

(Typical form of the output/response $y(t)$ to an input $g(t)$ in a linear eqn. w/ constant coefficients).

DEFINITION: For piecewise continuous functions f and g on $[0,\infty)$, the **convolution** of f and g is the function denoted $\boxed{f*g}$, defined by:

$$\boxed{(f*g)(t)=\int_0^t f(t-\tau)g(\tau)\,d\tau}$$

$$\boxed{=\int_0^t f(\tau)g(t-\tau)\,d\tau=(g*f)(t)}$$

(Convolution is commutative).

Example: $f(t)=e^t$; $g(t)=\sin t$

$$(f*g)(t)=\int_0^t e^{t-\tau}\sin\tau\,d\tau=e^t\int_0^t e^{-\tau}\sin\tau\,d\tau$$

$$=e^t\,\frac{1}{2}\left(-e^{-t}\sin t-e^{-t}\cos t+1\right)$$

$$\boxed{(f*g)(t)=\frac{1}{2}\left(e^t-\sin t-\cos t\right)}$$

$$I=\int_0^t e^{-\tau}\sin\tau\,d\tau=-\int_0^t (e^{-\tau})'\sin\tau\,d\tau$$

$$=-e^{-\tau}\sin\tau\Big|_0^t+\int_0^t e^{-\tau}\cos\tau\,d\tau$$

$$=-e^{-t}\sin t-\int_0^t (e^{-\tau})'\cos\tau\,d\tau$$

$$=-e^{-t}\sin t-e^{-\tau}\cos\tau\Big|_0^t-\underbrace{\int_0^t e^{-\tau}\sin\tau\,d\tau}_{I}$$

$$\Rightarrow 2I=-e^{-t}\sin t-e^{-t}\cos t+1$$

Convolution Theorem: If $f(t), g(t)$ are piecewise continuous and of exponential order on $t \in [0,\infty)$, then:

$$\boxed{\mathcal{L}\{f*g\} = F(s)G(s) = \mathcal{L}\{f(t)\}\,\mathcal{L}\{g(t)\}}$$

(Useful because it allows one to find the Laplace transform of a convolution without having to compute the convolution function).

Inverse Form:

$$\boxed{\mathcal{L}^{-1}\{F(s)G(s)\} = (f*g)(t)}$$

(Useful because it allows one to find inverse Laplace transforms of products $F(s)G(s)$ which are otherwise difficult to handle).

Example: $\mathcal{L}\left\{\int_0^t e^{\tau}\sin(t-\tau)\,d\tau\right\}$

$$= \mathcal{L}\{e^t * \sin t\}$$

$$= \mathcal{L}\{e^t\}\,\mathcal{L}\{\sin t\}$$

$$= \frac{1}{s-1}\,\frac{1}{s^2+1} = \frac{1}{(s-1)(s^2+1)}$$

Example: $\mathcal{L}^{-1}\left\{\frac{1}{(s^2+4)^2}\right\} = \mathcal{L}^{-1}\left\{F(s)G(s)\right\}$, where $F(s) = G(s) = \frac{1}{s^2+4}$

$$= (f*g)(t) \qquad \Rightarrow f(t) = g(t) = \frac{1}{2}\sin(2t)$$

$$\Rightarrow (f*g)(t) = \int_0^t \frac{1}{2}\sin(2\tau)\,\frac{1}{2}\sin\big(2(t-\tau)\big)\,d\tau$$

$$= \frac{1}{4}\int_0^t \sin(2\tau)\sin(2t-2\tau)\,d\tau \qquad \sin A\sin B = \frac{1}{2}\big(\cos(A-B) - \cos(A+B)\big)$$

$$= \frac{1}{8}\int_0^t \big(\cos(4\tau-2t) - \cos(2t)\big)\,d\tau$$

$$= \frac{1}{8}\left(\frac{1}{4}\sin(4\tau-2t)\Big|_{\tau=0}^{t} - t\cos(2t)\right)$$

$$= \frac{1}{8}\left(\frac{1}{4}\sin(2t) - \frac{1}{4}\sin(-2t) - t\cos(2t)\right)$$

$$= \frac{1}{8}\left(\frac{1}{2}\sin(2t) - t\cos(2t)\right) = \boxed{\frac{\sin(2t) - 2t\cos(2t)}{16}}$$

Example : $4y'' + y = g(t)$; $y(0) = 3$; $y'(0) = -7$.

$$\mathcal{L}\{4y'' + y\} = \mathcal{L}\{g(t)\}$$

$$4\left(s^2 Y(s) - s y(0) - y'(0)\right) + Y(s) = G(s)$$

$$(4s^2 + 1)\, Y(s) - 12s + 28 = G(s) \quad \Rightarrow \quad Y(s) = \frac{12s - 28}{4s^2 + 1} + \frac{G(s)}{4s^2 + 1}$$

$$= \frac{3s - 7}{s^2 + 1/4} + \frac{1}{4}\, \frac{1}{s^2 + 1/4}\, G(s)$$

$$\Rightarrow Y(s) = \mathcal{L}^{-1}\left\{\frac{3s - 7}{s^2 + 1/4}\right\} + \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s^2 + 1/4}\, G(s)\right\}$$

$$3\mathcal{L}^{-1}\left\{\frac{s}{s^2 + 1/4}\right\} - 7\mathcal{L}^{-1}\left\{\frac{1}{s^2 + 1/4}\right\}$$

$$= 3\cos(1/2\, t) - 14 \sin(1/2\, t)$$

$$\mathcal{L}^{-1}\{F(s)G(s)\} = (f * g)(t)$$

with $f(t) = 2\sin(1/2\, t)$

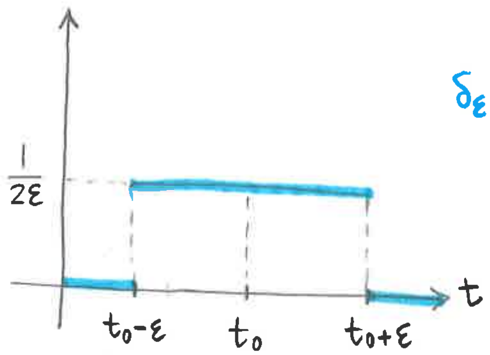
$$\Rightarrow \boxed{\begin{aligned} Y(s) &= 3\cos(1/2\, t) - 14\sin(1/2\, t) + \frac{1}{2}\left(\sin(1/2\, t) * g(t)\right) \\ &= 3\cos(1/2\, t) - 14\sin(1/2\, t) + \frac{1}{2}\int_0^t \sin\left(\frac{\tau}{2}\right) g(t - \tau)\, d\tau \end{aligned}}$$

Usefulness : This provides a general formula of the solution for an unspecified forcing function $g(t)$. So for a given $g(t)$, all that needs to be done is compute the convolution integral above.

The Dirac Delta Function

Unit Impulse: Model the situation of an external force of large magnitude acting on a system (mechanical, electrical, oscillatory etc.) only for a very short time. (a lightning strike; a golf ball given a sharp blow by a club etc.).

$$\delta_\varepsilon(t-t_0)=\begin{cases} 0, & 0\le t<t_0-\varepsilon \\ \frac{1}{2\varepsilon}, & t_0-\varepsilon\le t<t_0+\varepsilon \\ 0, & t\ge t_0+\varepsilon \end{cases} = \frac{1}{2\varepsilon}\left(u_{t_0-\varepsilon}(t)-u_{t_0+\varepsilon}(t)\right)$$



$\delta_\varepsilon(t-t_0)$: An impulse of magnitude $\frac{1}{2\varepsilon}$, acting for (2ε) units of time, centered at $t=t_0$.

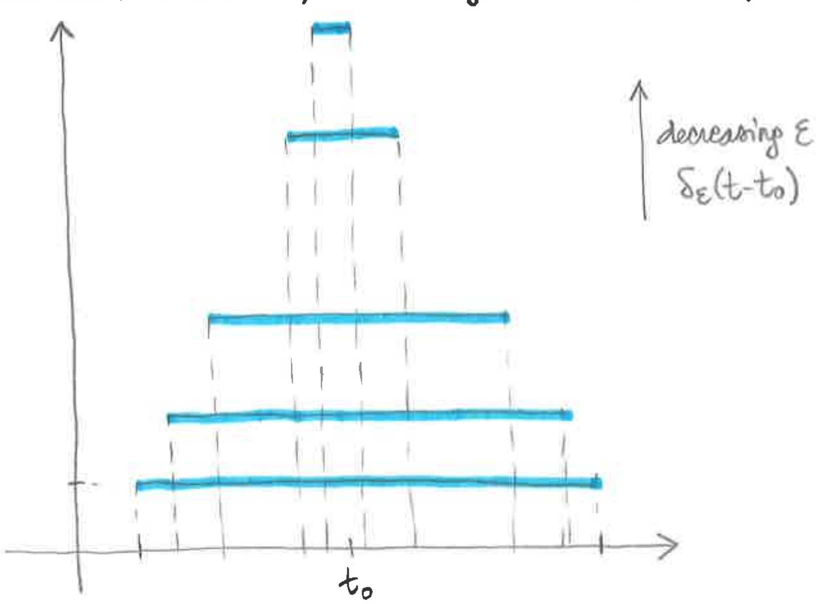
$$\int_0^\infty \delta_\varepsilon(t-t_0)\,dt = \frac{1}{2\varepsilon}(2\varepsilon)=1$$

$(\forall)\varepsilon>0$: $\boxed{\int_0^\infty \delta_\varepsilon(t-t_0)\,dt=1}$ (therefore the name "unit" impulse).

Dirac Delta Function: To model a strong external force acting instantaneously at $t=t_0$ we define:

$$\boxed{\delta(t-t_0)=\lim_{\varepsilon\to 0}\delta_\varepsilon(t-t_0)} \leftarrow \text{Dirac delta function}$$

Effectively, this lets the time interval $(t_0-\varepsilon, t_0+\varepsilon)$ around time t_0 decrease to 0. In the same time, the magnitude of the force, $\frac{1}{2\varepsilon}$, tends to ∞.



So in the limit, $\delta(t-t_0)$ is the "impulse":

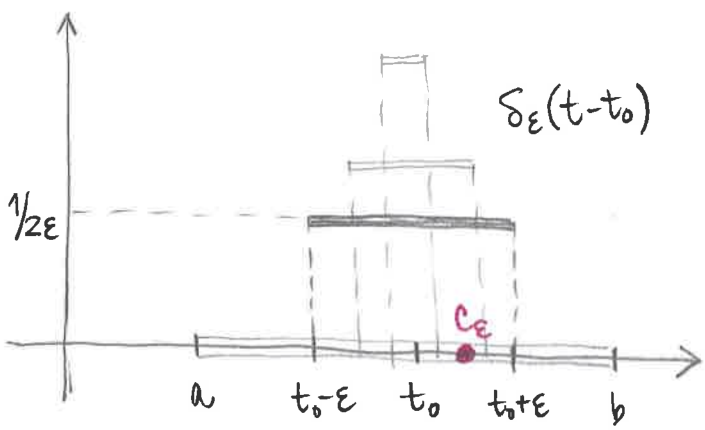
$$\boxed{\delta(t-t_0)=\begin{cases} 0, & \text{if } t\ne t_0 \\ \infty, & \text{if } t=t_0 \end{cases}}$$

As you can see, this is not really a "function". In fact, rigorously speaking, δ is a **distribution** (aka **generalized function**), an advanced concept which is the subject of its own branch of mathematics – **distribution theory**. This began with Laurent Schwartz trying to make sense of Dirac's strange "function".

Roughly speaking, a distribution is an operator acting on functions, defined by how it acts on functions. For the Dirac delta, the defining action is:

> **For any continuous function f on an interval $[a,b]$ containing a point t_0:**
>
> $$\boxed{\int_a^b f(t)\,\delta(t-t_0)\,dt = f(t_0)}$$

Proof.:



$$\int_a^b f(t)\,\delta_\varepsilon(t-t_0)\,dt =$$

$$= \int_{t_0-\varepsilon}^{t_0+\varepsilon} f(t)\,\frac{1}{2\varepsilon}\,dt$$

$$= \frac{1}{2\varepsilon}(t_0+\varepsilon-t_0+\varepsilon)\,f(c_\varepsilon)$$

$$= f(c_\varepsilon) \quad \text{for some } c_\varepsilon \in [t_0-\varepsilon, t_0+\varepsilon]$$

Mean Value Theorem for Integrals:

If f is continuous on $[a,b]$, then there is at least one value $c \in [a,b]$ such that:

$$\int_a^b f(x)\,dx = (b-a)\,f(c)$$

Letting $\varepsilon \to 0$,

$$\int_a^b f(t)\,\delta(t-t_0)\,dt = f(t_0)$$

since $\lim_{\varepsilon\to 0} f(c_\varepsilon) = t_0$

(this is forced by the interval $[t_0-\varepsilon, t_0+\varepsilon]$ collapsing onto the point $t=t_0$ as $\varepsilon \to 0$).

Some Important Special Cases of:

$$\boxed{\int_a^b f(t)\,\delta(t-t_0)\,dt = f(t_0)}$$

$a=0;\ b=\infty$

$$\boxed{\int_0^\infty f(t)\,\delta(t-t_0)\,dt = f(t_0)}$$

aka the "sifting" property, because $\delta(t-t_0)$ "sifts" through all the values of f on $[0,\infty)$ to yield $f(t_0)$.

Taking $f(t)=e^{-st}$ above:

$$\underbrace{\int_0^\infty e^{-st}\delta(t-t_0)\,dt}_{\mathcal{L}\{\delta(t-t_0)\}} = e^{-st_0}$$

$$\boxed{\mathcal{L}\{\delta(t-t_0)\} = e^{-st_0}}$$

In particular, letting $t_0=0$ above:

$$\boxed{\boxed{\mathcal{L}\{\delta(t)\}=1}}$$

This is another indication that δ is not really a function: recall that for a piecewise continuous f, of exponential order, we expect $\lim\limits_{s\to\infty} F(s)=0$.

$a=0;\ b=t_0$

$$\boxed{\int_0^{t_0} f(t)\,\delta(t_0-t)\,dt = f(t_0)}$$

This follows because δ is <u>even</u>, i.e. $\delta(t-t_0)=\delta(t_0-t)$.
But notice that this is really

$$(f*\delta)(t_0) = f(t_0).$$

In other words:

$$\boxed{\boxed{f*\delta = f}}$$

<u>δ is the "identity" wrt convolution.</u>

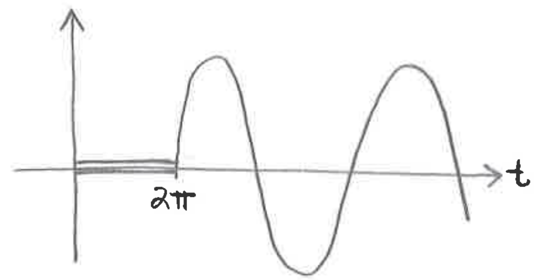
Example: $y'' + y = 4\delta(t-2\pi)$ → models the motion of a mass on a spring that is given a sharp blow at $t = 2\pi$.

$y(0) = 0;\ y'(0) = 0$ → mass begins at rest in the equilibrium position.

$$\mathcal{L}\{y''+y\} = \mathcal{L}\{4\delta(t-2\pi)\}$$

$$s^2Y(s) + Y(s) = 4e^{-2\pi s} \Rightarrow Y(s) = \frac{4e^{-2\pi s}}{s^2+1} \Rightarrow y(t) = 4\sin(t-2\pi)u_{2\pi}(t)$$
$$= 4\sin(t)u_{2\pi}(t)$$

$$\boxed{y(t) = 4\sin(t)u_{2\pi}(t) = \begin{cases} 0, & \text{if } 0 \le t < 2\pi \\ 4\sin t, & \text{if } t \geq 2\pi \end{cases}}$$



Example: $y'' + y = 4\delta(t-2\pi)$

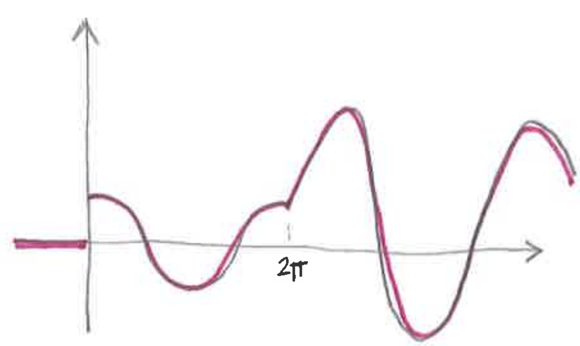
$y(0) = 1;\ y'(0) = 0$ → mass is released from rest 1 unit below the equilibrium point.

$$s^2Y(s) - sy(0) - y'(0) + Y(s) = 4e^{-2\pi s}$$

$$(s^2+1)Y(s) = s + 4e^{-2\pi s} \Rightarrow Y(s) = \frac{s}{s^2+1} + \frac{4e^{-2\pi s}}{s^2+1}$$

$$\Rightarrow \boxed{y(t) = \cos(t) + 4\sin(t)u_{2\pi}(t)}$$

$$\Rightarrow \boxed{y(t) = \begin{cases} \cos(t); & 0 \le t < 2\pi \\ \cos(t) + 4\sin(t); & t \geq 2\pi \end{cases}}$$



Remark: The condition $y'(0) = 0$ should be interpreted here in the sense that $\lim_{t \to 0_-} y'(t) = 0$.