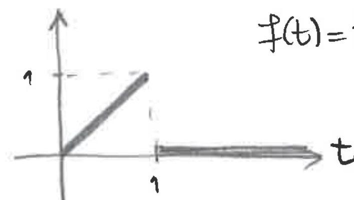
Homework 15

– Solutions –

1. $y'+2y=f(t)$; $y(0)=0$; $f(t)=\begin{cases} t, & 0\le t<1 \\ 0, & t\ge 1 \end{cases}$



$f(t)=t(1-u_1(t))$

$\mathcal{L}\{f(t)\}=\mathcal{L}\{t-tu_1(t)\}$

$=\frac{1}{s^2}-e^{-s}\left(\frac{1}{s^2}+\frac{1}{s}\right)$

$=\frac{1-e^{-s}-se^{-s}}{s^2}$

$\mathcal{L}\{y'+2y\}=\mathcal{L}\{f(t)\}$

$sY(s)-\underbrace{y(0)}_{0}+2Y(s)=\frac{1-e^{-s}-se^{-s}}{s^2}$

$\Rightarrow Y(s)=\frac{1-e^{-s}-se^{-s}}{s^2(s+2)}$

$\Rightarrow y(t)=\mathcal{L}^{-1}\left\{\frac{1-e^{-s}-se^{-s}}{s^2(s+2)}\right\}=$

$=\mathcal{L}^{-1}\left\{\frac{1}{s^2(s+2)}\right\}-\mathcal{L}^{-1}\left\{\frac{1}{s^2(s+2)}e^{-s}\right\}-\mathcal{L}^{-1}\left\{\frac{1}{s(s+2)}e^{-s}\right\}$

$\frac{1}{s^2(s+2)}=\frac{A}{s}+\frac{B}{s^2}+\frac{C}{s+2}$

$B=1/2$ $C=1/4$

$1=As(s+2)+\frac{1}{2}(s+2)+\frac{1}{4}s^2$

Match coeff. of s^2: $0=A+1/4$

$\Rightarrow A=-1/4$

$\boxed{\mathcal{L}^{-1}\left\{\frac{1}{s^2(s+2)}\right\}=-\frac{1}{4}+\frac{1}{2}t+\frac{1}{4}e^{-2t}}$

$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s+2)}\right\}\Big|_{t\to t-1}u_1(t)$

$=\left(-\frac{1}{4}+\frac{1}{2}(t-1)+\frac{1}{4}e^{-2(t-1)}\right)u_1(t)$

$\frac{1}{s(s+2)}=\frac{1}{2}\left(\frac{1}{s}-\frac{1}{s+2}\right)$

$\mathcal{L}^{-1}\left\{\frac{1}{s(s+2)}e^{-s}\right\}=\mathcal{L}^{-1}\left\{\frac{1}{s(s+2)}\right\}\Big|_{t\to t-1}u_1(t)$

$=\left(\frac{1}{2}-\frac{1}{2}e^{-2(t-1)}\right)u_1(t)$

$\Rightarrow y(t)=-\frac{1}{4}+\frac{1}{2}t+\frac{1}{4}e^{-2t}+\left(\frac{1}{4}-\frac{1}{2}(t-1)-\frac{1}{4}e^{-2(t-1)}-\frac{1}{2}+\frac{1}{2}e^{-2(t-1)}\right)u_1(t)$

$=\boxed{-\frac{1}{4}+\frac{1}{2}t+\frac{1}{4}e^{-2t}-\left(\frac{1}{4}+\frac{1}{2}(t-1)-\frac{1}{4}e^{-2t+2}\right)u_1(t)}$

(2) $y'' + 4y = (\sin t)\, u_{2\pi}(t)$; $y(0) = 1$; $y'(0) = 0$

$$\mathcal{L}\{y''\} + 4\mathcal{L}\{y\} = \mathcal{L}\{(\sin t)\, u_{2\pi}(t)\}$$

$$s^2 Y(s) - s y(0) - y'(0) + 4Y(s) = \mathcal{L}\{\sin(t-2\pi)\, u_{2\pi}(t)\}$$

$$(s^2+4)Y(s) - s = e^{-2\pi s}\mathcal{L}\{\sin t\}$$

$$(s^2+4)Y(s) = s + e^{-2\pi s}\frac{1}{1+s^2} \Rightarrow \boxed{Y(s) = \frac{s}{s^2+4} + \frac{1}{(s^2+1)(s^2+4)} e^{-2\pi s}}$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s^2+4)} e^{-2\pi s}\right\}$$

$$= \cos(2t) + \mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s^2+4)}\right\}\Big|_{t\to t-2\pi} u_{2\pi}(t)$$

$$\frac{1}{(s^2+1)(s^2+4)} = \frac{1}{3}\frac{(s^2+4)-(s^2+1)}{(s^2+1)(s^2+4)} = \frac{1}{3}\left(\frac{1}{s^2+1} - \frac{1}{s^2+4}\right)$$

$$= \cos(2t) + \frac{1}{3}\left(\sin t - \frac{1}{2}\sin(2t)\right)\Big|_{t\to t-2\pi} u_{2\pi}(t)$$

$$= \cos(2t) + \left(\frac{1}{3}\sin(t-2\pi) - \frac{1}{6}\sin(2t-4\pi)\right)u_{2\pi}(t)$$

$$\boxed{y(t) = \cos(2t) + \left(\frac{1}{3}\sin t - \frac{1}{6}\sin(2t)\right)u_{2\pi}(t)}$$

(3) $y'' + y = f(t)$; $y(0) = 0$; $y'(0) = 1$; $f(t) = \begin{cases} 0, & 0 \le t < \pi \\ 1, & \pi \le t < 2\pi \\ 0, & t \ge 2\pi \end{cases} = u_\pi(t) - u_{2\pi}(t).$

$$s^2 Y(s) - 1 + Y(s) = \mathcal{L}\{u_\pi(t) - u_{2\pi}(t)\}$$

$$(s^2+1)Y(s) - 1 = \frac{e^{-\pi s} - e^{-2\pi s}}{s} \Rightarrow Y(s) = \frac{e^{-\pi s}}{s(s^2+1)} - \frac{e^{-2\pi s}}{s(s^2+1)} + \frac{1}{s^2+1}$$

$$\frac{1}{s(s^2+1)} = \frac{(s^2+1)-s^2}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}$$

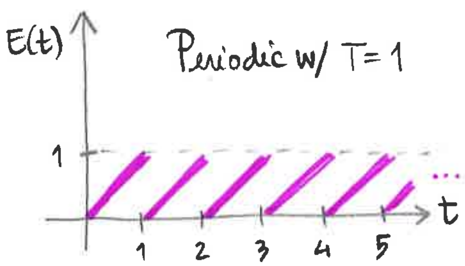
$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = 1 - \cos t$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\}\Big|_{t\to t-\pi} u_\pi(t) - \mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\}\Big|_{t\to t-2\pi} u_{2\pi}(t) + \sin t$$

$$= (1-\cos(t-\pi))u_\pi(t) - (1-\cos(t-2\pi))u_{2\pi}(t) + \sin t$$

$$\Rightarrow \boxed{y(t) = (1+\cos t)u_\pi(t) - (1-\cos t)u_{2\pi}(t) + \sin t}$$

④ Solve the L-R circuit equation $L\frac{di}{dt}+Ri=E(t)$ subject to $i(0)=0$, where $E(t)$ is given below:



$$\mathcal{L}\{E(t)\}=\frac{1}{1-e^{-s}}\int_0^1 e^{-st}f(t)\,dt$$
$$=\frac{1}{1-e^{-s}}\mathcal{L}\{t(1-u_1(t))\}$$
$$=\frac{1}{1-e^{-s}}\left(\mathcal{L}\{t\}-\mathcal{L}\{t\,u_1(t)\}\right)$$
$$=\frac{1}{1-e^{-s}}\left(\frac{1}{s^2}-\mathcal{L}\{((t-1)+1)u_1(t)\}\right)$$
$$=\frac{1}{1-e^{-s}}\left(\frac{1}{s^2}-e^{-s}\mathcal{L}\{t+1\}\right)$$
$$=\frac{1}{1-e^{-s}}\left(\frac{1}{s^2}-\frac{e^{-s}}{s^2}-\frac{e^{-s}}{s}\right)$$
$$=\frac{1}{1-e^{-s}}\;\frac{1-e^{-s}-se^{-s}}{s^2}$$

$$\Rightarrow L\mathcal{L}\{i'\}+R\mathcal{L}\{i\}=\mathcal{L}\{E(t)\}$$
$$\Rightarrow (Ls+R)I(s)=\frac{1-e^{-s}-se^{-s}}{s^2(1-e^{-s})}$$
$$\Rightarrow \boxed{I(s)=\frac{1-e^{-s}-se^{-s}}{s^2(Ls+R)(1-e^{-s})}}$$

$$\Rightarrow i(t)=\mathcal{L}^{-1}\left\{\frac{1-e^{-s}-se^{-s}}{s^2(Ls+R)(1-e^{-s})}\right\}=\mathcal{L}^{-1}\left\{\frac{1}{s^2(Ls+R)}\right\}-\mathcal{L}^{-1}\left\{\frac{1}{s(Ls+R)(1-e^{-s})}e^{-s}\right\} \quad (*)$$

$$\frac{1}{s^2(Ls+R)}=\frac{A}{s}+\frac{B}{s^2}+\frac{C}{Ls+R}\Rightarrow B=1/R \qquad C=L^2/R^2$$

$$1=As(Ls+R)+\frac{1}{R}(Ls+R)+\frac{L^2}{R^2}s^2$$

Match coeff. of s^2: $0=AL+\frac{L^2}{R^2}\Rightarrow A=-L/R^2$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{s^2(Ls+R)}\right\}=-\frac{L}{R^2}+\frac{1}{R}t+\frac{L}{R^2}e^{-(R/L)t}=\boxed{\frac{1}{R}t+\frac{L}{R^2}\left(e^{-Rt/L}-1\right)}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{S(LS+R)}\ \frac{e^{-S}}{1-e^{-S}}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{R}\left(\frac{1}{S}-\frac{1}{S+R/L}\right)\frac{e^{-S}}{1-e^{-S}}\right\}$$

$$\left.\begin{array}{l}\frac{1}{1-X} = \sum_{K=0}^{\infty} X^K;\ \forall\ |X|<1 \\ X=e^{-S};\ S>0\end{array}\right\} \Rightarrow \frac{1}{1-e^{-S}} = \sum_{K=0}^{\infty} e^{-SK} \Rightarrow \frac{e^{-S}}{1-e^{-S}} = \sum_{K=0}^{\infty} e^{-S(K+1)}$$

$$\rightarrow \boxed{\frac{e^{-S}}{1-e^{-S}} = \sum_{K=1}^{\infty} e^{-SK}}$$

$$\begin{aligned}
\Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{S(LS+R)}\ \frac{e^{-S}}{1-e^{-S}}\right\} &= \frac{1}{R}\ \mathcal{L}^{-1}\left\{\frac{1}{S}\sum_{K=1}^{\infty} e^{-SK}\right\} - \frac{1}{R}\mathcal{L}^{-1}\left\{\frac{1}{S+R/L}\ \sum_{K=1}^{\infty} e^{-SK}\right\} \\
&= \frac{1}{R}\sum_{K=1}^{\infty} \mathcal{L}^{-1}\left\{\frac{1}{S}e^{-SK}\right\} - \frac{1}{R}\sum_{K=1}^{\infty} \mathcal{L}^{-1}\left\{\frac{1}{S+R/L}e^{-SK}\right\} \\
&= \frac{1}{R}\sum_{K=1}^{\infty}\left(u_K(t) - e^{-R/L(t-K)}u_K(t)\right) \\
&= \boxed{\frac{1}{R}\sum_{K=1}^{\infty}\left(1-e^{-R/L(t-K)}\right)u_K(t)}
\end{aligned}$$

$\Rightarrow$ Back to (*):

$$\boxed{i(t) = \frac{1}{R}t + \frac{L}{R^2}\left(e^{-Rt/L}-1\right) + \frac{1}{R}\sum_{K=1}^{\infty}\left(e^{-R/L(t-K)}-1\right)u_K(t)}$$