Math 2552 - Differential Equations **Homework 15**
Sections F1 – F4; L1 – L4
Georgia Institute of Technology, Fall 2015

Laplace Transform:
ODEs with Discontinuous Forcing Functions

Solve the initial value problems below:

1. $y' + 2y = f(t);\ \ y(0) = 0;\ \ f(t) = \begin{cases} t, & 0 \le t < 1 \\ 0, & t \ge 1. \end{cases}$

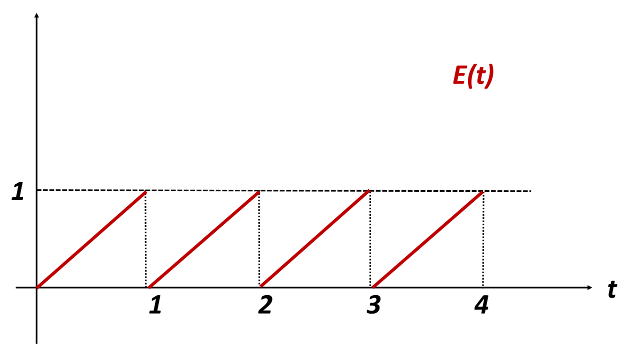
2. $y'' + 4y = (\sin t)u_{2\pi}(t);\ \ y(0) = 1;\ \ y'(0) = 0.$

3. $y'' + y = f(t);\ \ y(0) = 0;\ \ y'(0) = 1;\ \ f(t) = \begin{cases} 0, & 0 \le t < \pi \\ 1, & \pi \le t < 2\pi \\ 0, & t \ge 2\pi. \end{cases}$

4. Solve the L-R circuit equation

$$L\frac{di}{dt} + Ri = E(t),$$

for the current function $i(t)$, subject to $i(0) = 0$, where the voltage $E(t)$ is the periodic function given below:



You may need to use the following series expansion:

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k;\ \ |x| < 1.$$

Laplace transforms of some basic functions

$\mathscr{L}\{1\} = \frac{1}{s};\ \ s > 0$

$\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}};\ \ s > 0$

$\mathscr{L}\{e^{kt}\} = \frac{1}{s-k};\ \ s > k$

$\mathscr{L}\{\sin(kt)\} = \frac{k}{s^2+k^2};\ \ s > 0$

$\mathscr{L}\{\cos(kt)\} = \frac{s}{s^2+k^2};\ \ s > 0$

$\mathscr{L}\{u_a(t)\} = \frac{e^{-as}}{s};\ s > 0$

$\mathscr{L}\{\sinh(kt)\} = \frac{k}{s^2-k^2};\ \ s > |k|$

$\mathscr{L}\{\cosh(kt)\} = \frac{s}{s^2-k^2};\ \ s > |k|$

Inverse Laplace transforms of some basic functions

$\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} = 1$

$\mathscr{L}^{-1}\left\{\frac{1}{s^n}\right\} = \frac{1}{(n-1)!}t^{n-1}$

$\mathscr{L}^{-1}\left\{\frac{1}{s-k}\right\} = e^{kt}$

$\mathscr{L}^{-1}\left\{\frac{1}{s^2+k^2}\right\} = \frac{1}{k}\sin(kt)$

$\mathscr{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\} = \cos(kt)$

$\mathscr{L}^{-1}\left\{\frac{e^{-as}}{s}\right\} = u_a(t)$

$\mathscr{L}^{-1}\left\{\frac{1}{s^2-k^2}\right\} = \frac{1}{k}\sinh(kt)$

$\mathscr{L}^{-1}\left\{\frac{s}{s^2-k^2}\right\} = \cosh(kt)$

Properties of the Laplace and Inverse Laplace transform

Translation Theorem I:

$$\mathscr{L}\left\{e^{kt}f(t)\right\} = F(s-k) = \mathscr{L}\{f(t)\}\big|_{s\to s-k}$$

$$\mathscr{L}^{-1}\{F(s-k)\} = \mathscr{L}^{-1}\left\{F(s)\big|_{s\to s-k}\right\} = e^{kt}\mathscr{L}^{-1}\{F(s)\} = e^{kt}f(t)$$

Translation Theorem II:

$$\mathscr{L}\left\{f(t-a)u_a(t)\right\} = e^{-as}F(s) = e^{-as}\mathscr{L}\{f(t)\}$$

$$\mathscr{L}^{-1}\{e^{-as}F(s)\} = f(t-a)u_a(t) = \mathscr{L}^{-1}\{F(s)\}\big|_{t\to t-a}\,u_a(t)$$

Derivatives of Laplace Transforms:

$$\mathscr{L}\left\{t^n f(t)\right\} = (-1)^n\frac{d^n}{ds^n}F(s)$$

$$\mathscr{L}^{-1}\left\{F^{(n)}(s)\right\} = (-1)^n t^n f(t)$$

Laplace Transform of Periodic Functions:
If $f(t)$ is piecewise continuous on $[0,\infty)$, of exponential order, and periodic with period T:

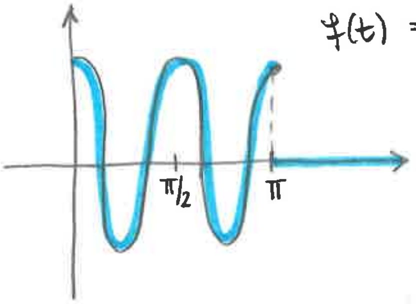
$$\mathscr{L}\{f(t)\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)\,dt$$

Laplace Transforms of Derivatives:

$$\begin{aligned}
\mathscr{L}\{y'\} &= sY(s) - y(0)\\
\mathscr{L}\{y''\} &= s^2Y(s) - sy(0) - y'(0)\\
\mathscr{L}\{y'''\} &= s^3Y(s) - s^2y(0) - sy'(0) - y''(0)\\
&\ \ \vdots\\
\mathscr{L}\{y^{(n)}(t)\} &= s^nY(s) - s^{n-1}y(0) - s^{n-2}y'(0) - \ldots - sy^{(n-2)}(0) - y^{(n-1)}(0)
\end{aligned}$$

ODEs with Discontinuous Forcing Functions

Example: $y'' + 16y = f(t)$; $y(0)=0$; $y'(0)=1$; $f(t) = \begin{cases} \cos(4t), & 0 \le t < \pi \\ 0, & t \ge \pi. \end{cases}$



$f(t)$ = external force acting on a mechanical system for only a short period of time, and is then removed.

$$f(t) = \cos(4t)\left(1 - u_\pi(t)\right)$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \mathcal{L}\{\cos(4t)\} - \mathcal{L}\{\cos(4t)\,u_\pi(t)\}$$

$$= \frac{s}{s^2+16} - \mathcal{L}\{\cos(4(t-\pi))\,u_\pi(t)\}$$

$$= \frac{s}{s^2+16} - e^{-\pi s}\mathcal{L}\{\cos(4t)\} = \frac{(1-e^{-\pi s})s}{s^2+16}$$

$$\Rightarrow \mathcal{L}\{y''\} + 16\mathcal{L}\{y\} = \mathcal{L}\{f(t)\}$$

$$s^2 Y(s) - s\underbrace{y(0)}_{0} - \underbrace{y'(0)}_{1} + 16\,Y(s) = \frac{(1-e^{-\pi s})s}{s^2+16}$$

$$\Rightarrow (s^2+16)Y(s) = 1 + \frac{s}{s^2+16} - \frac{e^{-\pi s}s}{s^2+16} \Rightarrow \boxed{Y(s) = \frac{1}{s^2+16} + \frac{s}{(s^2+16)^2} - \frac{s}{(s^2+16)^2}e^{-\pi s}}$$

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+16)^2}\right\} \ ?$$

Use: $\mathcal{L}^{-1}\{F^{(n)}(s)\} = (-1)^n t^n f(t)$

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+16)^2}\right\} = \mathcal{L}^{-1}\left\{\left(\frac{1}{s^2+16}\right)'\right\}\left(-\frac{1}{2}\right)$$

$$= -\frac{1}{2}(-1)\,t\,\mathcal{L}^{-1}\left\{\frac{1}{s^2+16}\right\}$$

$$= \frac{1}{2}t \cdot \frac{1}{4}\sin(4t)$$

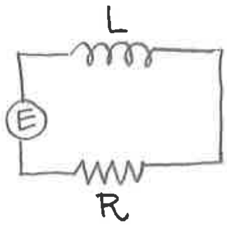
$$= \boxed{\frac{1}{8}t\sin(4t)}$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+16}\right\} + \mathcal{L}^{-1}\left\{\frac{s}{(s^2+16)^2}\right\} - \mathcal{L}^{-1}\left\{\frac{s}{(s^2+16)^2}\right\}\Big|_{t \to t-\pi} u_\pi(t)$$

$$= \frac{1}{4}\sin(4t) + \frac{1}{8}t\sin(4t) - \frac{1}{8}(t-\pi)\sin(4(t-\pi))\,u_\pi(t)$$

$$\Rightarrow \boxed{y(t) = \frac{1}{4}\sin(4t) + \frac{1}{8}t\sin(4t) - \frac{1}{8}(t-\pi)\sin(4t)\,u_\pi(t)}$$

Example : L-R Series Circuit



Series circuit containing a resistor R and an inductor L

$i(t)$ = current at time t.

$E(t)$ = voltage impressed on the circuit.

Kirchhoff's Second Law :

$L\dfrac{di}{dt}$ = voltage drop across the inductor

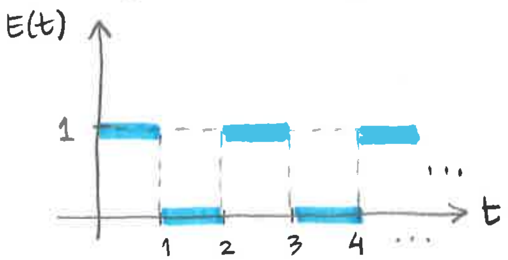
iR = voltage drop across the resistor

$$\boxed{L\frac{di}{dt} + Ri = E(t)}$$

→ a differential equation for the current $i(t)$.

↳ Solving this depends on how complicated the expression for voltage $E(t)$ is. Often times, $E(t)$ is discontinuous.

1 Find the current $i(t)$ in a single-loop L-R series circuit, given that the voltage $E(t)$ is the square wave below; and the initial condition $i(0)=0$.



$E(t)$ = periodic w/ period $T=2$

$$\Rightarrow \mathcal{L}\{E(t)\} = \frac{1}{1-e^{-2s}}\int_0^2 e^{-st}E(t)\,dt$$

$$= \frac{1}{1-e^{-2s}}\mathcal{L}\{1-u_1(t)\}$$

$$= \frac{1}{1-e^{-2s}}\left(\frac{1}{s}-\frac{e^{-s}}{s}\right) = \frac{1-e^{-s}}{s(1-e^{-2s})}$$

$$= \frac{1-e^{-s}}{s(1-e^{-s})(1+e^{-s})} = \frac{1}{s(1+e^{-s})}$$

Laplace Transform of ODE :

$$L\mathcal{L}\{i'\} + R\mathcal{L}\{i\} = \mathcal{L}\{E(t)\}$$

$$L(sI(s) - i(0)) + RI(s) = \mathcal{L}\{E(t)\}$$

$$(Ls+R)I(s) = \mathcal{L}\{E(t)\}$$

$$= \frac{1}{s(1+e^{-s})}$$

$$\Rightarrow \boxed{I(s) = \frac{1}{s(Ls+R)(1+e^{-s})}}$$

$$\Rightarrow \boxed{i(t) = \mathcal{L}^{-1}\left\{\frac{1}{s(Ls+R)(1+e^{-s})}\right\}}$$

How to find $\mathcal{L}^{-1}\left\{\frac{1}{s(Ls+R)(1+e^{-s})}\right\} \stackrel{?}{=} i(t)$?

- We can decompose $\frac{1}{s(Ls+R)} = \frac{A}{s} + \frac{B}{Ls+R}$ $\Rightarrow$ $\frac{1}{Ls+R}\Big|_{s=0} = A \Rightarrow A = 1/R$

$\Rightarrow \frac{1}{s}\Big|_{s=-R/L} = B \Rightarrow B = -L/R$

$$\frac{1}{s(Ls+R)} = \frac{1}{Rs} - \frac{L}{R(Ls+R)}$$

- So we now have

$$i(t) = \frac{1}{R}\mathcal{L}^{-1}\left\{\frac{1}{s}\,\frac{1}{1+e^{-s}}\right\} - \frac{1}{R}\mathcal{L}^{-1}\left\{\frac{1}{s+R/L}\,\frac{1}{1+e^{-s}}\right\}.$$

- Problem: we know how to handle things like $\mathcal{L}^{-1}\{e^{-as}F(s)\}$, but how to deal with the $\frac{1}{1+e^{-s}}$ factor? Series expansions!

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots = \sum_{k=0}^{\infty}(-1)^k x^k \quad \text{for all } x \in \mathbb{R} \text{ with } |x|<1.$$

Let $x = e^{-s}$ above, with $s>0$:

$$\Rightarrow \boxed{\frac{1}{1+e^{-s}} = \sum_{k=0}^{\infty}(-1)^k e^{-sk}}, \quad \forall\, s>0.$$

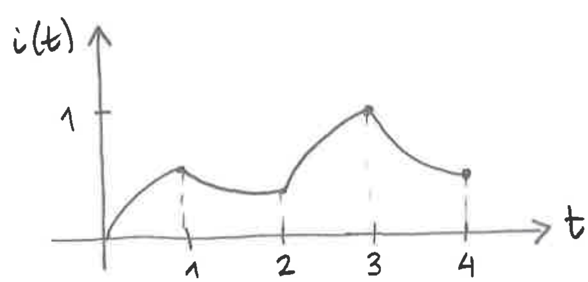
$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{s}\,\frac{1}{1+e^{-s}}\right\} = \sum_{k=0}^{\infty}(-1)^k\mathcal{L}^{-1}\left\{\frac{1}{s}e^{-sk}\right\} = \sum_{k=0}^{\infty}(-1)^k\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}\Big|_{t\to t-k} u_k(t)$$
$$= \sum_{k=0}^{\infty}(-1)^k u_k(t).$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{s+R/L}\,\frac{1}{1+e^{-s}}\right\} = \sum_{k=0}^{\infty}(-1)^k\mathcal{L}^{-1}\left\{\frac{1}{s+R/L}e^{-sk}\right\} = \sum_{k=0}^{\infty}(-1)^k\mathcal{L}^{-1}\left\{\frac{1}{s+R/L}\right\}\Big|_{t\to t-k} u_k(t)$$
$$= \sum_{k=0}^{\infty}(-1)^k e^{-R/L(t-k)}u_k(t)$$

$$\Rightarrow \boxed{i(t) = \frac{1}{R}\sum_{k=0}^{\infty}(-1)^k\left(1-e^{-R/L(t-k)}\right)u_k(t)}$$

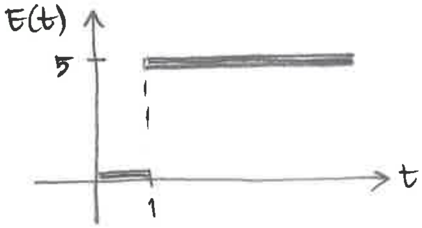
<u>Example</u>: Take $R=L=1$ and $0\le t<4$. Then

$$i(t) = \begin{cases} 1-e^{-t}; & 0\le t<1 \\ -e^{-t}+e^{-(t-1)}; & 1\le t<2 \\ 1-e^{-t}+e^{-(t-1)}-e^{-(t-2)}; & 2\le t<3 \\ -e^{-t}+e^{-(t-1)}-e^{-(t-2)}+e^{-(t-3)}; & 3\le t<4 \end{cases}$$



(2) Find the current $i(t)$ in a single-loop L-R circuit, where $L=R=1$, $i(0)=0$ and the voltage $E(t)$ is given by

$$E(t)=\begin{cases}0, & 0\le t<1\\ 5, & t\ge 1\end{cases}$$



$Li'+Ri=E(t)$ becomes $\boxed{i'+i=E(t)}$

$E(t)=5u_1(t)$

$\Rightarrow \mathcal{L}\{E(t)\}=\frac{5}{s}e^{-s}$

$\Rightarrow \mathcal{L}\{i'\}+\mathcal{L}\{i\}=\mathcal{L}\{E(t)\}$

$\Rightarrow sI(s)-i(0)+I(s)=\frac{5}{s}e^{-s}$

$\Rightarrow (s+1)I(s)=\frac{5}{s}e^{-s}$

$\Rightarrow I(s)=\frac{5}{s(s+1)}e^{-s}$

$$\Rightarrow i(t)=\mathcal{L}^{-1}\left\{\frac{5}{s(s+1)}e^{-s}\right\}=\mathcal{L}^{-1}\left\{\frac{5}{s(s+1)}\right\}\Big|_{t\to t-1}u_1(t)$$

$$\frac{1}{s(s+1)}=\frac{1}{s}-\frac{1}{s+1}$$

$$=5\left(1-e^{-t}\right)\Big|_{t\to t-1}u_1(t)$$

$$=5\left(1-e^{-t+1}\right)u_1(t).$$

$$\boxed{i(t)=5\left(1-e^{-t+1}\right)u_1(t)}$$