Unit Step Functions & Periodic Functions

Homework (14) - Solutions

(1) $\mathcal{L}\{(t-1)u_1(t)\} = e^{-s}\mathcal{L}\{t\} = \frac{1}{s^2}e^{-s}.$

(2) $\mathcal{L}\{t\,u_2(t)\} = \mathcal{L}\{(t-2+2)u_2(t)\} = \mathcal{L}\{(t-2)u_2(t)\} + 2\mathcal{L}\{u_2(t)\}$

$$= e^{-2s}\mathcal{L}\{t\} + 2\frac{e^{-2s}}{s} = e^{-2s}\left(\frac{1}{s^2}+\frac{2}{s}\right).$$

(3) $\mathcal{L}\{\cos(2t)u_\pi(t)\} = \mathcal{L}\{\cos(2t-2\pi)u_\pi(t)\} = \mathcal{L}\{\cos(2(t-\pi))u_\pi(t)\}$

$$= e^{-\pi s}\mathcal{L}\{\cos(2t)\} = e^{-\pi s}\frac{s}{s^2+4}.$$

(4) $\mathcal{L}\{(t-1)^3e^{t-1}u_1(t)\} = e^{-s}\mathcal{L}\{t^3e^t\} = e^{-s}\left(\mathcal{L}\{t^3\}\Big|_{s\to s-1}\right) = e^{-s}\left(\frac{3!}{s^4}\Big|_{s\to s-1}\right)$

$$= \frac{6e^{-s}}{(s-1)^4}$$

(5) $\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^3}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\}\Big|_{t\to t-2}u_2(t) = \left(\frac{1}{2!}t^2\Big|_{t\to t-2}\right)u_2(t) = \frac{(t-2)^2}{2}u_2(t).$

(6) $\mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{s^2+1}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}\Big|_{t\to t-\pi}u_\pi(t) = \sin(t-\pi)u_\pi(t) = -\sin t\cdot u_\pi(t).$

(7) $\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s(s+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s(s+1)}\right\}\Big|_{t\to t-1}u_1(t) = (1-e^{-t+1})u_1(t).$

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}-\frac{1}{s+1}\right\} = 1-e^{-t}$$

(8) $\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}e^{-3s}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\}\Big|_{t\to t-3}u_3(t) = \cos(2t-6)u_3(t)$

(9) $f(t) = \begin{cases} 2, & 0\le t<3 \\ -2, & t\ge 3\end{cases}$

$$f(t) = 2(1-u_3(t)) - 2u_3(t) = 2-4u_3(t)$$

$$\mathcal{L}\{f(t)\} = \frac{2}{s} - \frac{4}{s}e^{-3s}$$

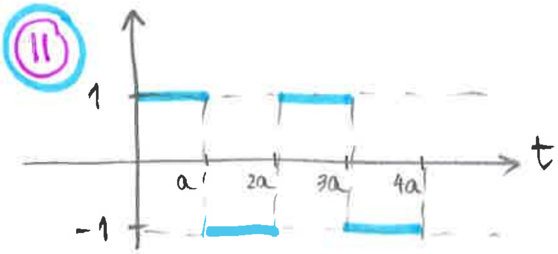
(10) $f(t) = \begin{cases} 0, & 0\le t<1 \\ t^2, & t\ge 1\end{cases}$

$$f(t) = t^2u_1(t)$$

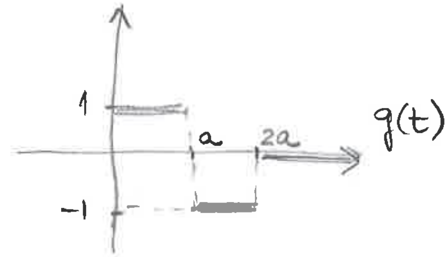
$$\mathcal{L}\{f(t)\} = \mathcal{L}\{t^2u_1(t)\} = \mathcal{L}\{[(t-1)+1]^2u_1(t)\} = \mathcal{L}\{[(t-1)^2+2(t-1)+1]u_1(t)\}$$

$$= e^{-s}\mathcal{L}\{t^2\} + 2e^{-s}\mathcal{L}\{t\} + \mathcal{L}\{u_1(t)\} = e^{-s}\left(\frac{2}{s^3}+\frac{2}{s^2}+\frac{1}{s}\right).$$

11



Period $T = 2a$



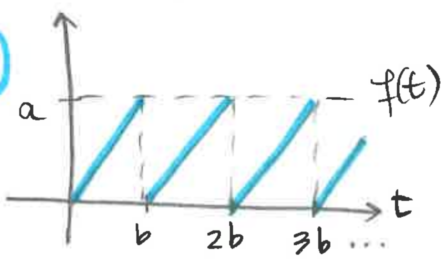
$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-2as}} \underbrace{\int_0^{2a} e^{-st} f(t)\,dt}_{\mathcal{L}\{g(t)\}}$$

$$g(t) = (1 - u_a(t)) - 1\,(u_a(t) - u_{2a}(t))$$
$$= 1 - 2u_a(t) + u_{2a}(t)$$
$$\mathcal{L}\{g(t)\} = \frac{1}{s} - 2\frac{e^{-as}}{s} + \frac{e^{-2as}}{s}$$

$$= \frac{1}{1-e^{-2as}}\left(\frac{1-2e^{-as}+e^{-2as}}{s}\right)$$

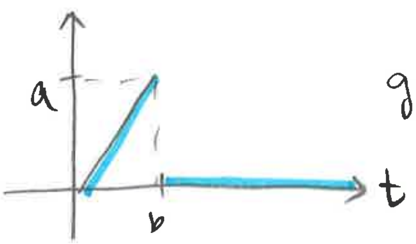
$$= \frac{(1-e^{-as})^2}{s(1-e^{-as})(1+e^{-as})} = \frac{1-e^{-as}}{s(1+e^{-as})}$$

12



Period $T = b$

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-bs}} \underbrace{\int_0^{b} e^{-st} f(t)\,dt}_{\mathcal{L}\{g(t)\}}$$



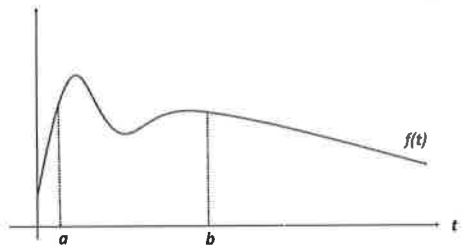
$$g(t) = \frac{a}{b}t\,(1 - u_b(t))$$

$$\mathcal{L}\{g(t)\} = \frac{a}{b}\mathcal{L}\{t - t\,u_b(t)\}$$
$$= \frac{a}{b}\left(\frac{1}{s^2} - \mathcal{L}\{(t-b)u_b(t)\} - b\,\mathcal{L}\{u_b(t)\}\right)$$
$$= \frac{a}{b}\left(\frac{1}{s^2} - e^{-bs}\frac{1}{s^2} - \frac{be^{-bs}}{s}\right)$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{1}{1-e^{-bs}}\,\frac{a}{b}\,e^{-bs}\left(e^{bs}\frac{1}{s^2} - \frac{1}{s^2} - \frac{b}{s}\right)$$

$$= \frac{1}{e^{bs}-1}\,\frac{a}{b}\,\frac{e^{bs}-1-bs}{s^2} = \boxed{\frac{a(e^{bs}-1-bs)}{bs^2(e^{bs}-1)}}$$

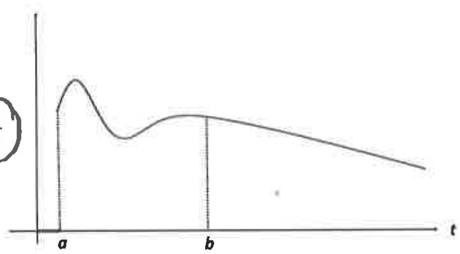
13. Consider the function $f(t)$, for $t \in [0, \infty)$, graphed below, and points $a, b \in [0, \infty)$.

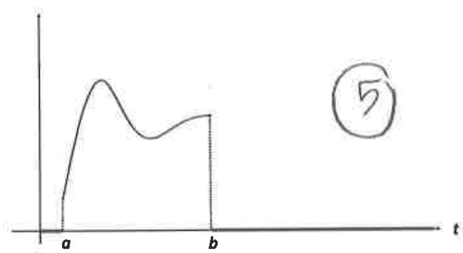


Match each of the following graphs (obtained by various translations and "turning off" of the graph of f) with the expressions below:

1. $f(t)(1 - u_b(t))$
2. $f(t)(u_a(t) - u_b(t))$
3. $f(t)(1 - u_a(t))$
4. $f(t)u_a(t)$
5. $f(t-a)(u_a(t) - u_b(t))$
6. $f(t-b)u_b(t)$

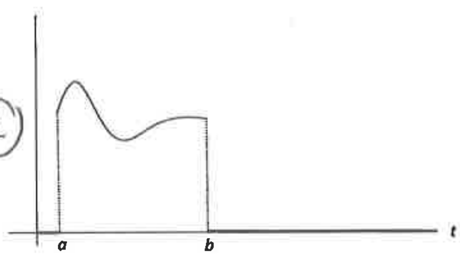
4

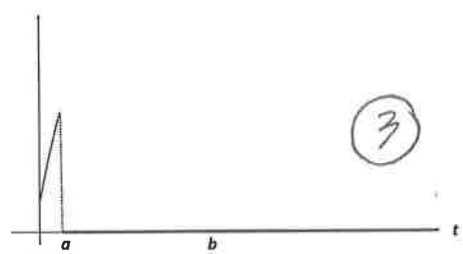




5

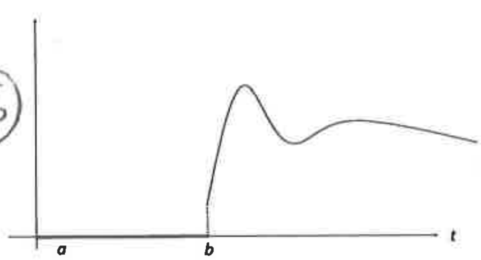
2

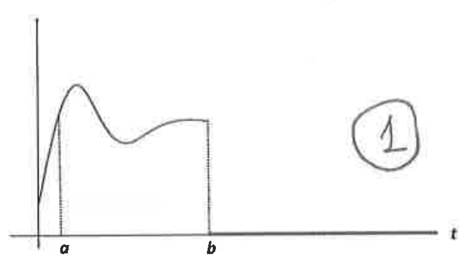




3

6





1