Math 2552 - Differential Equations **Homework 14**
Sections F1 – F4; L1 – L4
Georgia Institute of Technology, Fall 2015

Laplace Transform:
Unit Step Functions and Periodic Functions

Compute the following:

1. $\mathscr{L}\{(t-1)u_1(t)\}$

2. $\mathscr{L}\{tu_2(t)\}$

3. $\mathscr{L}\{\cos(2t)u_\pi(t)\}$

4. $\mathscr{L}\{(t-1)^3e^{t-1}u_1(t)\}$

5. $\mathscr{L}^{-1}\left\{\dfrac{e^{-2s}}{s^3}\right\}$

6. $\mathscr{L}^{-1}\left\{\dfrac{e^{-\pi s}}{s^2+1}\right\}$

7. $\mathscr{L}^{-1}\left\{\dfrac{e^{-s}}{s(s+1)}\right\}$

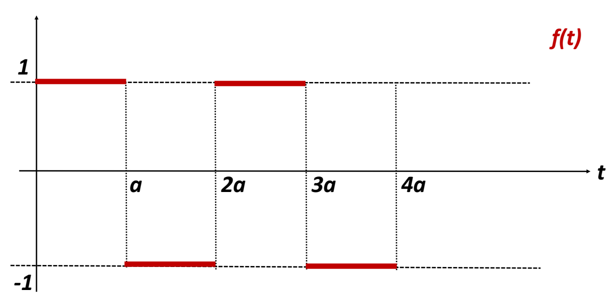
8. $\mathscr{L}^{-1}\left\{\dfrac{e^{-3s}s}{s^2+4}\right\}$

Express the following functions in terms of unit step functions and compute their Laplace transforms:

9. $f(t) = \begin{cases} 2 & ,0 \le t < 3 \\ -2 & ,t \ge 3. \end{cases}$

10. $f(t) = \begin{cases} 0 & ,0 \le t < 1 \\ t^2 & ,t \ge 1. \end{cases}$

11. The following graph describes the periodic function $f(t)$. Find $\mathscr{L}\{f(t)\}$.



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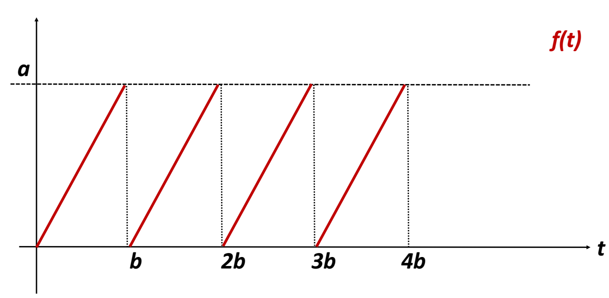
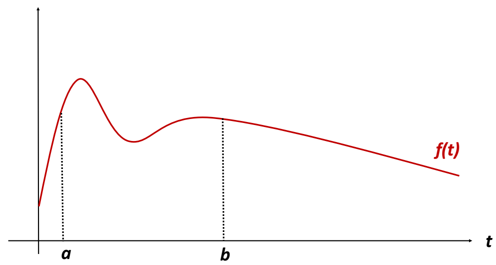
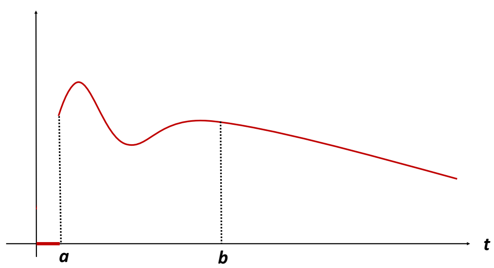


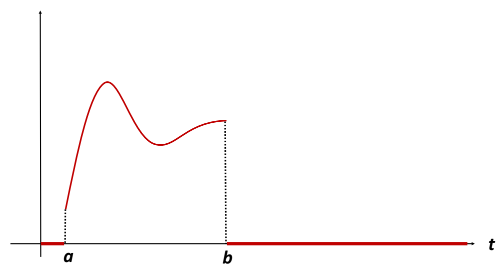
13. Consider the function $f(t)$, for $t \in [0, \infty)$, graphed below, and points $a, b \in [0, \infty)$.

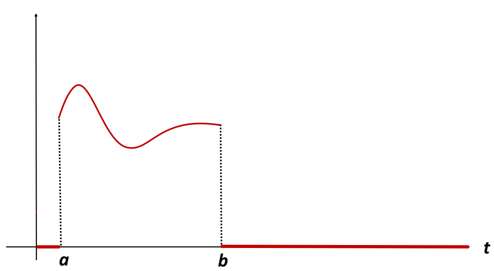


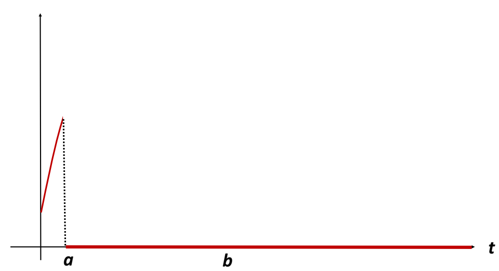
Match each of the following graphs (obtained by various translations and "turning off" of the graph of f) with the expressions below:

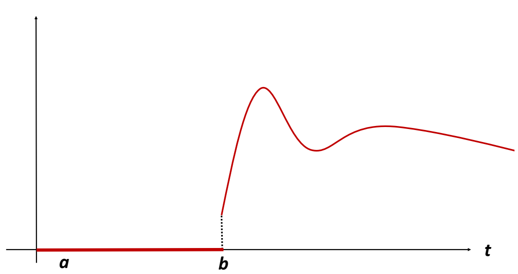
1. $f(t)(1 - u_b(t))$
2. $f(t)(u_a(t) - u_b(t))$
3. $f(t)(1 - u_a(t))$
4. $f(t)u_a(t)$
5. $f(t-a)(u_a(t) - u_b(t))$
6. $f(t-b)u_b(t)$

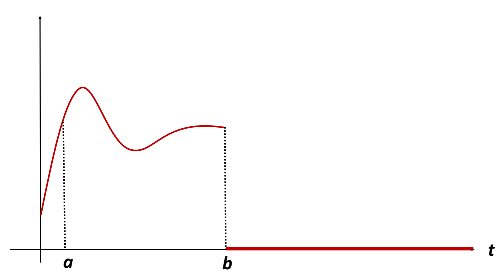












Laplace transforms of some basic functions

$$\mathscr{L}\{1\} = \frac{1}{s};\ \ s > 0 \qquad \mathscr{L}\{\sin(kt)\} = \frac{k}{s^2+k^2};\ \ s > 0 \qquad \mathscr{L}\{\sinh(kt)\} = \frac{k}{s^2-k^2};\ \ s > |k|$$

$$\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}};\ \ s > 0 \qquad \mathscr{L}\{\cos(kt)\} = \frac{s}{s^2+k^2};\ \ s > 0 \qquad \mathscr{L}\{\cosh(kt)\} = \frac{s}{s^2-k^2};\ \ s > |k|$$

$$\mathscr{L}\{e^{kt}\} = \frac{1}{s-k};\ \ s > k \qquad \mathscr{L}\{u_a(t)\} = \frac{e^{-as}}{s};\ s > 0$$

Inverse Laplace transforms of some basic functions

$$\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} = 1 \qquad \mathscr{L}^{-1}\left\{\frac{1}{s^2+k^2}\right\} = \frac{1}{k}\sin(kt) \qquad \mathscr{L}^{-1}\left\{\frac{1}{s^2-k^2}\right\} = \frac{1}{k}\sinh(kt)$$

$$\mathscr{L}^{-1}\left\{\frac{1}{s^n}\right\} = \frac{1}{(n-1)!}t^{n-1} \qquad \mathscr{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\} = \cos(kt) \qquad \mathscr{L}^{-1}\left\{\frac{s}{s^2-k^2}\right\} = \cosh(kt)$$

$$\mathscr{L}^{-1}\left\{\frac{1}{s-k}\right\} = e^{kt} \qquad \mathscr{L}^{-1}\left\{\frac{e^{-as}}{s}\right\} = u_a(t)$$

Properties of the Laplace and Inverse Laplace transform

Translation Theorem I:

$$\mathscr{L}\left\{e^{kt}f(t)\right\} = F(s-k) = \mathscr{L}\{f(t)\}\big|_{s\to s-k}$$

$$\mathscr{L}^{-1}\{F(s-k)\} = \mathscr{L}^{-1}\left\{F(s)\big|_{s\to s-k}\right\} = e^{kt}\mathscr{L}^{-1}\{F(s)\} = e^{kt}f(t)$$

Translation Theorem II:

$$\mathscr{L}\{f(t-a)u_a(t)\} = e^{-as}F(s) = e^{-as}\mathscr{L}\{f(t)\}$$

$$\mathscr{L}^{-1}\{e^{-as}F(s)\} = f(t-a)u_a(t) = \mathscr{L}^{-1}\{F(s)\}\big|_{t\to t-a}\,u_a(t)$$

Derivatives of Laplace Transforms:

$$\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}F(s)$$

$$\mathscr{L}^{-1}\left\{F^{(n)}(s)\right\} = (-1)^n t^n f(t)$$

Laplace Transform of Periodic Functions:
If $f(t)$ is piecewise continuous on $[0,\infty)$, of exponential order, and periodic with period T:

$$\mathscr{L}\{f(t)\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)\,dt$$

Laplace Transforms of Derivatives:

$$\begin{aligned}
\mathscr{L}\{y'\} &= sY(s) - y(0)\\
\mathscr{L}\{y''\} &= s^2Y(s) - sy(0) - y'(0)\\
\mathscr{L}\{y'''\} &= s^3Y(s) - s^2y(0) - sy'(0) - y''(0)\\
&\ \vdots\\
\mathscr{L}\{y^{(n)}(t)\} &= s^nY(s) - s^{n-1}y(0) - s^{n-2}y'(0) - \ldots - sy^{(n-2)}(0) - y^{(n-1)}(0)
\end{aligned}$$

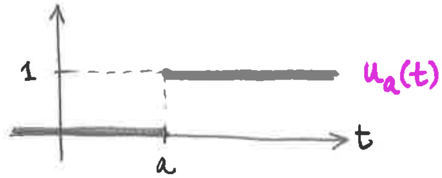
Laplace Transform & the Unit Step Function

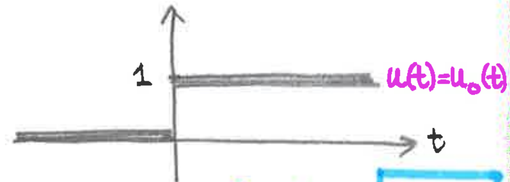
Def.: The Unit Step Function (aka the **Heaviside function**):

$$u(t) = \begin{cases} 0, & t<0 \\ 1, & t\geq 0 \end{cases}$$

- models situations where a signal can either be "on" or "off".
- Translations of $u(t)$ allows one to turn signals off at times other than 0:

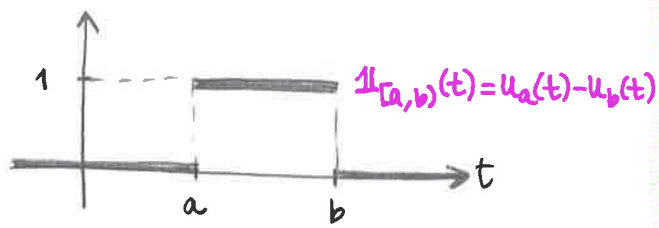
$$u_a(t) = \begin{cases} 0, & t<a \\ 1, & t\geq a \end{cases}$$



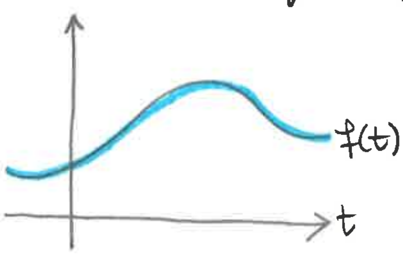


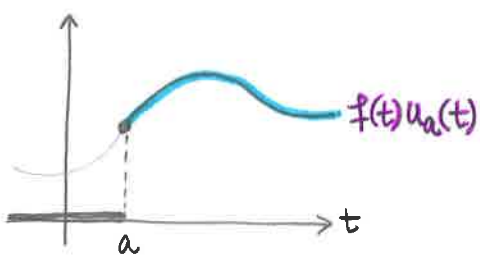
- We can turn a signal "on" at $t=a$ and "off" at $t=b$, using the **indicator function** $\mathbb{1}_{[a,b)}$ on $[a,b)$: "on" on $[a,b)$, "off" elsewhere:

$$\mathbb{1}_{[a,b)}(t) = u_a(t) - u_b(t) = \begin{cases} 0, & t<a \\ 1, & a\leq t<b \\ 0, & t>b \end{cases}$$

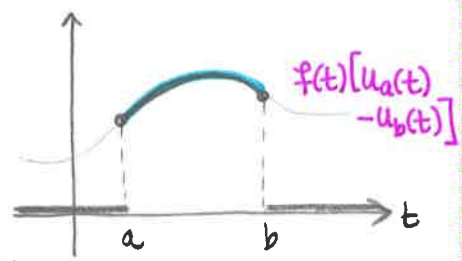


- Multiply a function $f(t)$ by $u_a(t)$ or $\mathbb{1}_{[a,b)}(t)$ to "turn off" portions of its graph: (also makes it easy to express piecewise-defined functions).





("turns off" the graph of f before $t=a$)



("turns off" the graph of f everywhere outside $a\leq t<b$).

- $$\mathcal{L}\{u_a(t)\} = \frac{e^{-as}}{s}\ ;\ s>0$$

$$\mathcal{L}\{u_a(t)\} = \int_a^\infty e^{-st}\,dt = -\frac{1}{s}e^{-st}\Big|_{t=a}^{\infty} = \frac{e^{-as}}{s}\ ;\ s>0.$$

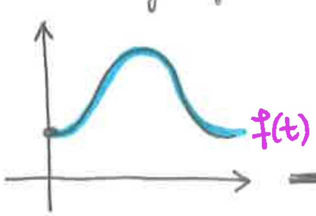
The Second Translation Theorem:

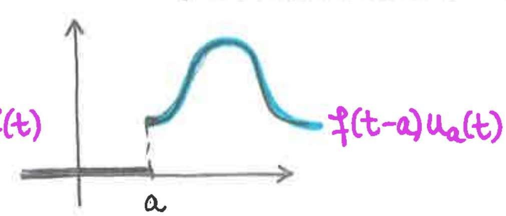
$(a>0)$ $$\mathcal{L}\{f(t-a)u_a(t)\} = e^{-as}F(s)$$

Inverse Form:

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)u_a(t)$$ $(a>0)$

Remark: What does $f(t-a)u_a(t)$ do? Say we have a function $f(t)$ defined on $[0,\infty)$. Translating by $a>0$, i.e. $f(t-a)$ "shifts" the graph of f to the right by a units, and is now a function defined on $[a,\infty)$. Multiplying by $u_a(t)$ does not change this, but only defines the new function to be 0 on $[0,a)$:





Proof: $\mathcal{L}\{f(t-a)u_a(t)\} = \int_0^\infty e^{-st} f(t-a)\underbrace{u_a(t)}_{\substack{0 \text{ before } t=a \\ 1 \text{ after}}} dt = \underbrace{\int_a^\infty e^{-st} f(t-a)\,dt}_{\text{change of variable: } t'=t-a}$

$$= \int_0^\infty e^{-s(t'+a)} f(t')\,dt' = e^{-as} \underbrace{\int_0^\infty e^{-st'} f(t')\,dt'}_{\mathcal{L}\{f(t)\}=F(s).}$$

$\Rightarrow dt' = dt$
$\Rightarrow t=a \Rightarrow t'=0$
$\Rightarrow t\to\infty \Rightarrow t'\to\infty$

Ex: $f(t) = \begin{cases} (t-2)^3, & t \geq 2 \\ 0, & 0 \leq t < 2 \end{cases}$

Express in terms of step functions:

$f(t) = (t-2)^3 u_2(t)$

$\Rightarrow \mathcal{L}\{f(t)\} = \mathcal{L}\{(t-2)^3 u_2(t)\} = e^{-2s}\mathcal{L}\{t^3\} = \frac{6}{s^4}e^{-2s}$

Ex: $f(t) = \begin{cases} (t-2)^3, & t \geq 3 \\ 0, & 0 \leq t < 3 \end{cases}$

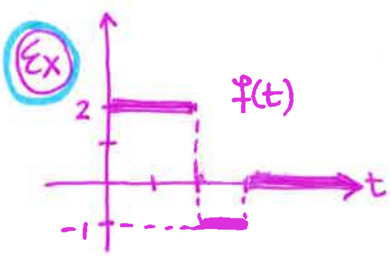
$f(t) = (t-2)^3 u_3(t)$

The Translation Thm. does not immediately apply – the $u_3(t)$ suggests it needs things in terms of $\underline{(t-3)}$:

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{(t-2)^3 u_3(t)\} = \mathcal{L}\{[(t-3)+1]^3 u_3(t)\} = e^{-3s}\mathcal{L}\{(t+1)^3\}$$

$$= e^{-3s}\mathcal{L}\{t^3+3t^2+3t+1\} = e^{-3s}\left(\frac{6}{s^4}+\frac{6}{s^3}+\frac{3}{s^2}+\frac{1}{s}\right).$$

Ex



Express in terms of step functions:

$$f(t) = 2\big(\underbrace{u_0(t)-u_2(t)}_{\mathbb{1}_{[0,2)}}\big) - 1\big(\underbrace{u_2(t)-u_3(t)}_{\mathbb{1}_{[2,3)}}\big)$$

$$= 2u_0(t) - 3u_2(t) + u_3(t)$$

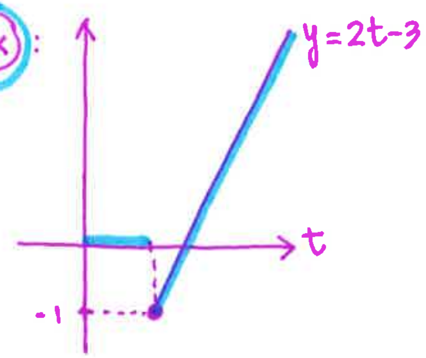
$$\Rightarrow \mathcal{L}\{f(t)\} = 2\mathcal{L}\{u_0(t)\} - 3\mathcal{L}\{u_2(t)\} + \mathcal{L}\{u_3(t)\}$$

$$= 2\cdot\frac{1}{s} - 3\cdot\frac{e^{-2s}}{s} + \frac{e^{-3s}}{s}.$$

Ex: $\mathcal{L}\{\sin(t)u_{2\pi}(t)\} = \mathcal{L}\{\sin(t-2\pi)u_{2\pi}(t)\} = e^{-2\pi s}\mathcal{L}\{\sin t\} = \frac{e^{-2\pi s}}{s^2+1}.$

(sin is periodic w/ period 2π).

Ex:



$f(t)$ = the line $y=2t-3$ "turned off" on $[0,1)$:

$$f(t) = (2t-3)u_1(t)$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \mathcal{L}\{(2t-3)u_1(t)\} = \mathcal{L}\{(2(t-1)-1)u_1(t)\}$$

$$= e^{-s}\mathcal{L}\{2t-1\} = e^{-s}\left(2\mathcal{L}\{t\} - \mathcal{L}\{1\}\right)$$

$$= e^{-s}\left(\frac{4}{s^2}-\frac{1}{s}\right).$$

Ex: $\mathcal{L}^{-1}\left\{\dfrac{e^{-\pi s/2}}{s^2+9}\right\} = \mathcal{L}^{-1}\left\{e^{-\frac{\pi}{2}s}\dfrac{1}{s^2+9}\right\}$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\}\Big|_{t\to t-\pi/2} u_{\pi/2}(t) = \frac{1}{3}\sin(3t)\Big|_{t\to t-\pi/2} u_{\pi/2}(t)$$

$$= \frac{1}{3}\sin(3t - 3\pi/2)\, u_{\pi/2}(t)$$

$$= \frac{1}{3}\cos(3t)\, u_{\pi/2}(t).$$

$\sin(3t - 3\pi/2) =$
$= \sin(3t - 3\pi/2 + 2\pi)$
$= \sin(3t + \pi/2) = \cos(3t)$

Laplace Transform of Periodic Functions

Periodic function f with period T: $f(t) = f(t+T),\ \forall t>0.$

Laplace transforms of periodic functions can be obtained by integration over 1 period:

> Suppose $f(t)$ is piecewise continuous on $t\in[0,\infty)$, of exponential order, and periodic with period T. Then
>
> $$\boxed{\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\,dt} = \boxed{\frac{1}{1-e^{-sT}}\mathcal{L}\{f(t)(1-u_T(t))\}}$$

Proof.: $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = \int_0^T e^{-st} f(t)\,dt + \int_T^\infty e^{-st} f(t)\,dt$

$t = u+T$, $dt = du$: $\int_T^\infty e^{-st} f(t)\,dt = \int_0^\infty e^{-s(u+T)} f(u+T)\,du$

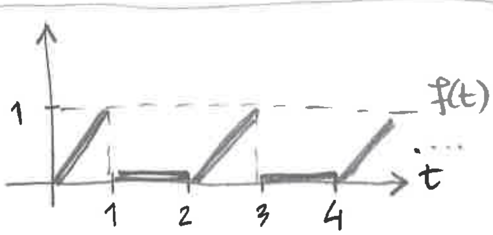
$= e^{-sT}\int_0^\infty e^{-su} f(u)\,du = e^{-sT}\mathcal{L}\{f(t)\}$

$\Rightarrow (1-e^{-sT})\mathcal{L}\{f(t)\} = \int_0^T e^{-st} f(t)\,dt$

$$= \int_0^\infty e^{-st} f(t)(1-u_T(t))\,dt$$

$$= \mathcal{L}\{f(t)(1-u_T(t))\}.$$

Ex:



Period: $T=2$

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-2s}}\int_0^2 e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2s}}\int_0^1 e^{-st}\, t\,dt$$

$$= \frac{1}{1-e^{-2s}}\mathcal{L}\{t(1-u_1(t))\}$$

$$= \frac{1-e^{-s}-se^{-s}}{(1-e^{-2s})s^2}$$

$\mathcal{L}\{t(1-u_1(t))\} = \mathcal{L}\{t\} - \mathcal{L}\{t u_1(t)\}$

$$= \frac{1}{s^2} - \mathcal{L}\{(t-1)u_1(t)\} - \mathcal{L}\{u_1(t)\}$$

$$= \frac{1}{s^2} - e^{-s}\frac{1}{s^2} - e^{-s}\frac{1}{s} = \frac{1-e^{-s}-se^{-s}}{s^2}$$