

The Inverse Laplace Transform

Find the inverse Laplace transforms below. (See next pages for useful formulas, examples etc.)

- | | | |
|---|---|--|
| 1. $\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\}$ | 9. $\mathcal{L}^{-1}\left\{\frac{2s-6}{s^2+9}\right\}$ | 17. $\mathcal{L}^{-1}\left\{\frac{s}{(s^2+4)(s+2)}\right\}$ |
| 2. $\mathcal{L}^{-1}\left\{\frac{1}{s^2}-\frac{48}{s^5}\right\}$ | 10. $\mathcal{L}^{-1}\left\{\frac{1}{s^2+3s}\right\}$ | 18. $\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^3}\right\}$ |
| 3. $\mathcal{L}^{-1}\left\{\frac{1}{s^2}-\frac{1}{s}+\frac{1}{s-2}\right\}$ | 11. $\mathcal{L}^{-1}\left\{\frac{s}{s^2+2s-3}\right\}$ | 19. $\mathcal{L}^{-1}\left\{\frac{s}{s^2+4s+5}\right\}$ |
| 4. $\mathcal{L}^{-1}\left\{\frac{(s+1)^3}{s^4}\right\}$ | 12. $\mathcal{L}^{-1}\left\{\frac{3s}{s+4}\right\}$ | 20. $\mathcal{L}^{-1}\left\{\frac{1}{s^2-6s+10}\right\}$ |
| 5. $\mathcal{L}^{-1}\left\{\frac{5}{s^2+49}\right\}$ | 13. $\mathcal{L}^{-1}\left\{\frac{s}{(s-2)(s-3)(s-6)}\right\}$ | 21. $\mathcal{L}^{-1}\left\{\frac{s}{(s+1)^2}\right\}$ |
| 6. $\mathcal{L}^{-1}\left\{\frac{1}{s^2-16}\right\}$ | 14. $\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+4)}\right\}$ | 22. $\mathcal{L}^{-1}\left\{\frac{2s-1}{s^2(s+1)^3}\right\}$ |
| 7. $\mathcal{L}^{-1}\left\{\frac{1}{4s+1}\right\}$ | 15. $\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s^2+4)}\right\}$ | 23. $\mathcal{L}^{-1}\left\{\frac{s^3-2s^2-6s-6}{(s^2+2s+2)s}\right\}$ |
| 8. $\mathcal{L}^{-1}\left\{\frac{4s}{4s^2+1}\right\}$ | 16. $\mathcal{L}^{-1}\left\{\frac{2s+4}{(s-2)(s^2+4s+3)}\right\}$ | 24. $\mathcal{L}^{-1}\left\{\frac{s^3-2s^2-6s-6}{(s^2+2s+2)s^2}\right\}$ |

25. The inverse Laplace transform *may not be unique*. To see this, compute the Laplace transform of:

$$f(t) = \begin{cases} 1, & t \in [0, 1) \cup (1, 2) \cup (2, \infty) \\ 3, & t = 1 \\ 4, & t = 2 \end{cases}$$

and check that it is the same as $\mathcal{L}\{1\}$.

Solve the IVPs below *using the Laplace transform*:

26. $y' - 3y = e^{2t}$; $y(0) = 1$.
27. $y'' - 6y' + 9y = t^2 e^{3t}$; $y(0) = 2, y'(0) = 6$.
28. $y'' + 4y' + 6y = 1 + e^{-t}$; $y(0) = y'(0) = 0$.
29. $y'' - y' = e^t \cos t$; $y(0) = y'(0) = 0$.

Laplace transforms of some basic functions

$$\begin{aligned} \mathcal{L}\{1\} &= \frac{1}{s}; \quad s > 0 & \mathcal{L}\{\sin(kt)\} &= \frac{k}{s^2 + k^2}; \quad s > 0 & \mathcal{L}\{\sinh(kt)\} &= \frac{k}{s^2 - k^2}; \quad s > |k| \\ \mathcal{L}\{t^n\} &= \frac{n!}{s^{n+1}}; \quad s > 0 & \mathcal{L}\{\cos(kt)\} &= \frac{s}{s^2 + k^2}; \quad s > 0 & \mathcal{L}\{\cosh(kt)\} &= \frac{s}{s^2 - k^2}; \quad s > |k| \\ \mathcal{L}\{e^{kt}\} &= \frac{1}{s - k}; \quad s > k \end{aligned}$$

Inverse Laplace transforms of some basic functions

$$\begin{aligned} \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} &= 1 & \mathcal{L}^{-1}\left\{\frac{1}{s^2 + k^2}\right\} &= \frac{1}{k} \sin(kt) & \mathcal{L}^{-1}\left\{\frac{1}{s^2 - k^2}\right\} &= \frac{1}{k} \sinh(kt) \\ \mathcal{L}^{-1}\left\{\frac{1}{s^n}\right\} &= \frac{1}{(n-1)!} t^{n-1} & \mathcal{L}^{-1}\left\{\frac{s}{s^2 + k^2}\right\} &= \cos(kt) & \mathcal{L}^{-1}\left\{\frac{s}{s^2 - k^2}\right\} &= \cosh(kt) \\ \mathcal{L}^{-1}\left\{\frac{1}{s - k}\right\} &= e^{kt} \end{aligned}$$

Properties of the Laplace and Inverse Laplace transform

Translation Theorem:

$$\mathcal{L}\{e^{kt} f(t)\} = F(s - k) = \mathcal{L}\{f(t)\}|_{s \rightarrow s - k}$$

$$\mathcal{L}^{-1}\{F(s - k)\} = \mathcal{L}^{-1}\{F(s)|_{s \rightarrow s - k}\} = e^{kt} f(t)$$

Derivatives of Laplace Transforms:

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)$$

$$\mathcal{L}^{-1}\{F^{(n)}(s)\} = (-1)^n t^n f(t)$$

Laplace Transforms of Derivatives:

$$\begin{aligned} \mathcal{L}\{y'\} &= sY(s) - y(0) \\ \mathcal{L}\{y''\} &= s^2Y(s) - sy(0) - y'(0) \\ \mathcal{L}\{y'''\} &= s^3Y(s) - s^2y(0) - sy'(0) - y''(0) \\ &\vdots \\ \mathcal{L}\{y^{(n)}(t)\} &= s^n Y(s) - s^{n-1}y(0) - s^{n-2}y'(0) - \dots - sy^{(n-2)}(0) - y^{(n-1)}(0) \end{aligned}$$

• INVERSE LAPLACE TRANSFORM •

• Laplace transform: $f(t) \xrightarrow{\mathcal{L}} F(s) = \mathcal{L}\{f(t)\}$

• Inverse problem: given $F(s)$, find $f(t)$ such that $F(s) = \mathcal{L}\{f(t)\} \Rightarrow f(t) = \mathcal{L}^{-1}\{F(s)\}$

$$f(t) = \mathcal{L}^{-1}\{F(s)\} \xleftarrow{\mathcal{L}^{-1}} F(s)$$

Inverse Laplace Transforms we already know:

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+k^2}\right\} = \frac{1}{k} \sin(kt)$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2-k^2}\right\} = \frac{1}{k} \sinh(kt)$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^n}\right\} = \frac{1}{(n-1)!} t^{n-1}$$

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\} = \cos(kt)$$

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2-k^2}\right\} = \cosh(kt)$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s-k}\right\} = e^{kt}$$

Basic Examples:

$$\mathcal{L}^{-1}\left\{\frac{1}{s^5}\right\} = \frac{1}{4!} t^4 ; \quad \mathcal{L}^{-1}\left\{\frac{3}{s^2+64}\right\} = \frac{3}{8} \sin(8t) ; \quad \mathcal{L}^{-1}\left\{\frac{5}{s^2-16}\right\} = \frac{5}{4} \sinh(4t).$$

$$\mathcal{L}^{-1}\left\{\frac{3}{s}\right\} = 3 ; \quad \mathcal{L}^{-1}\left\{\frac{2s}{s^2+64}\right\} = 2 \cos(8t) ; \quad \mathcal{L}^{-1}\left\{\frac{\sqrt{2}s}{s^2-7}\right\} = \sqrt{2} \cosh(\sqrt{7}t).$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} = e^{-2t} ; \quad \mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} = e^{3t} ; \quad \mathcal{L}^{-1}\left\{\frac{2s-1}{s^2+16}\right\} = 2 \cos(4t) - \frac{1}{4} \sin(4t).$$

TRANSLATION THEOREM (Inverse Laplace Form):

$$\mathcal{L}^{-1}\{F(s-k)\} = \mathcal{L}^{-1}\{F(s)|_{s \rightarrow s-k}\} = e^{kt} f(t) ; \quad f(t) = \mathcal{L}^{-1}\{F(s)\}$$

Examples:

$$\textcircled{\bullet} \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^3}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^3} \Big|_{s \rightarrow s+2}\right\} = e^{-2t} \mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\} = \frac{1}{2} e^{-2t} \cdot t^2.$$

$$\textcircled{\bullet} \mathcal{L}^{-1}\left\{\frac{s}{s^2+6s+11}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{(s+3)^2+2}\right\} = \mathcal{L}^{-1}\left\{\frac{s+3-3}{(s+3)^2+2}\right\}$$

[Completing the square]

$$= \mathcal{L}^{-1}\left\{\frac{s+3}{(s+3)^2+2}\right\} - 3 \mathcal{L}^{-1}\left\{\frac{1}{(s+3)^2+2}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s}{s^2+2} \Big|_{s \rightarrow s+3}\right\} - 3 \mathcal{L}^{-1}\left\{\frac{1}{s^2+2} \Big|_{s \rightarrow s+3}\right\}$$

$$= e^{-3t} \cos(\sqrt{2}t) - \frac{3}{\sqrt{2}} e^{-3t} \sin(\sqrt{2}t).$$

BEHAVIOR OF $F(s)$ AS $s \rightarrow \infty$: If $f(t)$ is a piecewise continuous function of exponential order on $t \in [0, \infty)$, then

$$\lim_{s \rightarrow \infty} [\mathcal{L}\{f(t)\}] = 0$$

\Rightarrow The Laplace transforms of $F_1(s) = s^2$ and $F_2(s) = \frac{s}{s+1}$ do not exist b/c $\lim_{s \rightarrow \infty} F_1(s) = \infty$; $\lim_{s \rightarrow \infty} F_2(s) = 1$.

\mathcal{L}^{-1} USING PARTIAL FRACTIONS

Look at 3 cases: $\left\{ \begin{array}{l} \text{Distinct Linear Factors: } \frac{1}{(s-1)(s+2)(s+4)} \\ \text{Repeated Linear Factors: } \frac{s+1}{s(s+2)^3} \\ \text{Irreducible Quadratic Factors: } \frac{3s-2}{s(s^2+4)} \end{array} \right.$

① Distinct Linear Factors: $F(s) = \frac{1}{(s-1)(s+2)(s+4)}$

$$F(s) = \frac{1}{(s-1)(s+2)(s+4)} = \frac{A}{s-1} + \frac{B}{s+2} + \frac{C}{s+4}$$

$$\Rightarrow \mathcal{L}^{-1}\{F(s)\} = A \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + B \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} + C \mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\}$$

? $\underbrace{\hspace{1cm}}_{e^t}$? $\underbrace{\hspace{1cm}}_{e^{-2t}}$? $\underbrace{\hspace{1cm}}_{e^{-4t}}$

$$\frac{1}{(s-1)(s+2)(s+4)} = \frac{A}{s-1} + \frac{B}{s+2} + \frac{C}{s+4}$$

Bring to the same denominator:

$$1 = A(s+2)(s+4) + B(s-1)(s+4) + C(s-1)(s+2)$$

Give convenient values to s :

$$s=1 \Rightarrow 1 = 15A \Rightarrow A = \frac{1}{15}$$

$$s=-2 \Rightarrow 1 = -6B \Rightarrow B = -\frac{1}{6}$$

$$s=-4 \Rightarrow 1 = 10C \Rightarrow C = \frac{1}{10}$$

"Cover up" Method:

- Cover up the term $(s-1)$ and evaluate at $s=1$:

$$\frac{1}{\boxed{(s-1)}(s+2)(s+4)} \Big|_{s=1} = A \Rightarrow A = \frac{1}{15}$$

- Same for the other terms:

$$\frac{1}{(s-1)\boxed{(s+2)}(s+4)} \Big|_{s=-2} = B \Rightarrow B = -\frac{1}{6}$$

$$\frac{1}{(s-1)(s+2)\boxed{(s+4)}} \Big|_{s=-4} = C \Rightarrow C = \frac{1}{10}$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{(s-1)(s+2)(s+4)}\right\} = \frac{1}{15}e^t - \frac{1}{6}e^{-2t} + \frac{1}{10}e^{-4t}$$

② Repeated Linear Factors: $F(s) = \frac{3s^2 - 16s + 21}{(s-1)^2(s+3)}$

$$F(s) = \frac{3s^2 - 16s + 21}{(s-1)^2(s+3)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s+3}$$

$$\Rightarrow \mathcal{L}^{-1}\{F(s)\} = A \underbrace{\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}}_{e^t} + B \underbrace{\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\}}_{e^t \cdot t} + C \underbrace{\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\}}_{e^{-3t}}$$

$$\boxed{\frac{3s^2 - 16s + 21}{(s-1)^2(s+3)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s+3}}$$

Problem: When repeated factors occur, giving the "convenient" values to s does not give you all the coefficients.

$$3s^2 - 16s + 21 = A(s-1)(s+3) + B(s+3) + C(s-1)^2 \quad \left. \frac{3s^2 - 16s + 21}{(s-1)^2(s+3)} \right|_{s=1} = B \Rightarrow B = \frac{8}{4} = 2$$

$$s=1 \Rightarrow 8 = 4B \Rightarrow B=2$$

$$s=-3 \Rightarrow 96 = 16C \Rightarrow C=6$$

$$\left. \frac{3s^2 - 16s + 21}{(s-1)^2(s+3)} \right|_{s=-3} = C \Rightarrow C = \frac{96}{16} = 6.$$

To find A , you can for example:

⊙ Match coefficients of a power of s :

Ex: s^2 coefficients: $3 = A + C = A + 6 \Rightarrow A = -3$

or s coefficients: $-16 = 2A + B - 2C = 2A - 10 \Rightarrow 2A = -6 \Rightarrow A = -3$

or s^0 coefficients: $21 = -3A + 3B + C = -3A + 12 \Rightarrow -3A = 9 \Rightarrow A = -3.$

⊙ Give some other value to s :

Ex: $s=0 \Rightarrow 21 = -3A + 3B + C$ (essentially the same as matching the coefficients of s^0).

$$\mathcal{L}^{-1}\left\{\frac{3s^2 - 16s + 21}{(s-1)^2(s+3)}\right\} = -3e^t + 2te^t + 6e^{-3t}$$

③ Irreducible Quadratic Factor: $F(s) = \frac{3s}{(s^2+2)(s-1)}$

$$F(s) = \frac{3s}{(s^2+2)(s-1)} = \frac{As+B}{s^2+2} + \frac{C}{s-1}$$

$$\Rightarrow \mathcal{L}^{-1}\{F(s)\} = A \mathcal{L}^{-1}\left\{\frac{s}{s^2+2}\right\} + B \mathcal{L}^{-1}\left\{\frac{1}{s^2+2}\right\} + C \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}$$

? $\underbrace{\hspace{10em}}_{\cos(\sqrt{2}t)}$? $\underbrace{\hspace{10em}}_{\frac{1}{\sqrt{2}} \sin(\sqrt{2}t)}$? $\underbrace{\hspace{10em}}_{e^t}$

$$\boxed{\frac{3s}{(s^2+2)(s-1)} = \frac{As+B}{s^2+2} + \frac{C}{s-1}}$$

Problem: An irreducible factor has no "convenient" values to give s.

$$\boxed{3s = (As+B)(s-1) + C(s^2+2)}$$

Convenient value: $s=1 \Rightarrow 3 = 3C \Rightarrow C=1$ $\left| \frac{3s}{(s^2+2)(s-1)} \right|_{s=1} = C \Rightarrow C = \frac{3}{3} = 1.$

To find the remaining coefficients **A & B**, you can:

① Come up with a 2x2 linear equations in A & B. You can do this by

② Matching coefficients of powers of s:

s^2 coefficients: $0 = A + C = A + 1 \Rightarrow A = -1$

s^0 coefficients: $0 = -B + 2C = -B + 2 \Rightarrow B = 2$

③ Giving other values to s:

$s=0 \Rightarrow 0 = -B + 2C \Rightarrow B = 2$

$s=2 \Rightarrow 6 = (2A+B) + 6C \Rightarrow 6 = 2A + 2 + 6 \Rightarrow A = -1.$

④ Subtract the terms you know. That is, once you know $C=1$:

$$\frac{3s}{(s^2+2)(s-1)} = \frac{As+B}{s^2+2} + \frac{1}{s-1}$$

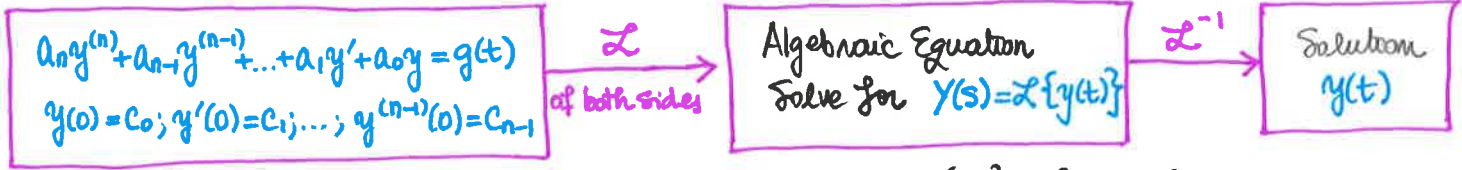
$$\Rightarrow \frac{As+B}{s^2+2} = \frac{3s}{(s^2+2)(s-1)} - \frac{1}{s-1} = \frac{3s - s^2 - 2}{(s^2+2)(s-1)} = -\frac{s^2 - 3s + 2}{(s^2+2)(s-1)} = -\frac{(s-2)(s-1)}{(s^2+2)(s-1)}$$

This equality should tell you that (s-1) has to be a factor of (3s - s^2 - 2)

$$\Rightarrow \frac{As+B}{s^2+2} = \frac{-s+2}{s^2+2} \Rightarrow \text{You can just determine } \boxed{A=-1; B=2}$$

by inspection at this point.

● Laplace Transform & Linear ODEs w/ Constant Coefficients (IVP) ●



$\mathcal{L}\{y'\} = sY(s) - y(0)$; $\mathcal{L}\{y''\} = s^2Y(s) - sy(0) - y'(0)$; $\mathcal{L}\{y'''\} = s^3Y(s) - s^2y(0) - sy'(0) - y''(0)$

Example: $y'' - 4y' + 4y = t^3 e^{2t}$; $y(0) = y'(0) = 0$ solved using all 3 methods so far.

Laplace Solution: $\mathcal{L}\{y'' - 4y' + 4y\} = \mathcal{L}\{t^3 e^{2t}\}$

$s^2 Y(s) - sy(0) - y'(0) - 4(sY(s) - y(0)) + 4Y(s) = \mathcal{L}\{t^3\}$ | $s \rightarrow s-2$

$(s^2 - 4s + 4)Y(s) = \frac{3!}{(s-2)^4} \Rightarrow Y(s) = \frac{6}{(s-2)^4}$

$\Rightarrow y(t) = \mathcal{L}^{-1}\left\{\frac{6}{(s-2)^4}\right\} = 6 \mathcal{L}^{-1}\left\{\frac{1}{s^4}\right\}_{s \rightarrow s-2} = 6 \cdot e^{2t} \frac{1}{3!} t^3$

$\Rightarrow y(t) = \frac{1}{20} t^5 e^{2t}$

Variation of Parameters:

① y_c : $w^2 - 4w + 4 = 0 \Rightarrow (w-2)^2 = 0$
 $y_c = c_1 e^{2t} + c_2 t e^{2t}$

② y_p :
 $W = \begin{vmatrix} e^{2t} & t e^{2t} \\ 2e^{2t} & (2t+1)e^{2t} \end{vmatrix} = e^{4t}$

$W_1 = \begin{vmatrix} 0 & t e^{2t} \\ t^3 e^{2t} & (2t+1)e^{2t} \end{vmatrix} = -t^4 e^{4t}$

$\Rightarrow u_1' = -t^4 \Rightarrow u_1 = -\frac{1}{5} t^5$

$W_2 = \begin{vmatrix} e^{2t} & 0 \\ 2e^{2t} & t^3 e^{2t} \end{vmatrix} = t^3 e^{4t}$

$\Rightarrow u_2' = t^3 \Rightarrow u_2 = \frac{1}{4} t^4$

$\Rightarrow y_p = -\frac{1}{5} t^5 e^{2t} + \frac{1}{4} t^4 e^{2t} = \frac{1}{20} t^5 e^{2t}$

③ $y = c_1 e^{2t} + c_2 t e^{2t} + \frac{1}{20} t^5 e^{2t}$

④ Find c_1, c_2 : $y(0) = 0 \Rightarrow c_1 = 0$
 $y'(0) = 0 \Rightarrow c_2 = 0$

$\Rightarrow y(t) = \frac{1}{20} t^5 e^{2t}$

Undetermined Coefficients: (use the y_c to the right)

$y_p = (At^5 + Bt^4 + Ct^3 + Dt^2) e^{2t}$

(Initial guess $y_p = (At^3 + Bt^2 + Ct + D) e^{2t}$ has to be multiplied by t^2 to eliminate duplication of y_c .)

$\Rightarrow y_p' = (2At^5 + (5A+2B)t^4 + (4B+2C)t^3 + (3C+2D)t^2 + 2Dt) e^{2t}$

$\Rightarrow y_p'' = (4At^5 + (20A+4B)t^4 + (20A+16B+4C)t^3 + (12B+12C+4D)t^2 + (6C+8D)t + 2D) e^{2t}$

Put back into original eqn. and match coefficients:

$t^5 e^{2t}$: $4A - 8A + 4A = 0$ ✓

$t^4 e^{2t}$: $20A + 4B - 20A - 8B + 4B = 0$ ✓

$t^3 e^{2t}$: $20A + 16B + 4C - 16B - 8C + 4C = 1$
 $\Rightarrow 20A = 1 \Rightarrow A = 1/20$

$t^2 e^{2t}$: $12B + 12C + 4D - 12C - 8D + 4D = 0$
 $\Rightarrow 12B = 0 \Rightarrow B = 0$

$t e^{2t}$: $6C + 8D - 8D = 0 \Rightarrow 6C = 0 \Rightarrow C = 0$

e^{2t} : $2D = 0 \Rightarrow D = 0$

$\Rightarrow y_p = \frac{1}{20} t^5 e^{2t}$

(Go back and add y_c , find $c_1 = c_2 = 0$).

What is the conclusion?

It is often easier to solve the type of problem:

Linear ODE w/ constant coeff. IVP w/ conditions at 0

Using the Laplace transform. It gets straight to the point and just becomes an \mathcal{L}^{-1} problem. Keyword here is IVP - if you want the general solution, you can use Laplace to maybe find y_p , but you must still find y_c .