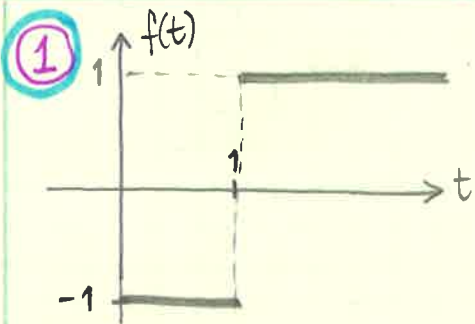


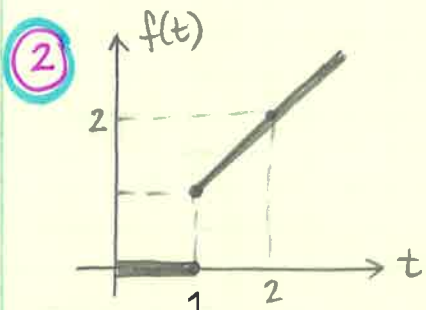
Laplace Transform: Definition

Homework (11)



$$\mathcal{L}\{f(t)\} = \frac{2}{s}e^{-s} - \frac{1}{s}; s > 0$$

$$\begin{aligned}\mathcal{L}\{f(t)\} &= \int_0^{\infty} e^{-st} f(t) dt \\ &= \int_0^1 e^{-st} \cdot (-1) dt + \int_1^{\infty} e^{-st} \cdot 1 dt \\ &= \frac{1}{s} e^{-st} \Big|_{t=0}^1 - \frac{1}{s} e^{-st} \Big|_{t=1}^{\infty} \\ &= \left(\frac{1}{s} e^{-s} - \frac{1}{s} \right) - \left(0 - \frac{1}{s} e^{-s} \right) \\ &= \frac{2}{s} e^{-s} - \frac{1}{s} \quad [s > 0]\end{aligned}$$



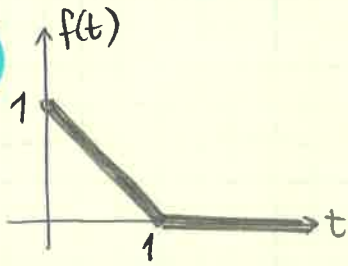
$$f(t) = \begin{cases} 0, & 0 \leq t < 1 \\ t, & 1 \leq t < \infty \end{cases}$$

$$\begin{aligned}\mathcal{L}\{f(t)\} &= \int_1^{\infty} e^{-st} \cdot t dt \\ &= \left(-\frac{t}{s} e^{-st} - \frac{1}{s^2} e^{-st} \right) \Big|_{t=1}^{\infty} \\ &= \frac{1}{s} e^{-s} + \frac{1}{s^2} e^{-s} \quad [s > 0]\end{aligned}$$

$$\mathcal{L}\{f(t)\} = \frac{s+1}{s^2} e^{-s}; s > 0$$

$$\begin{aligned}\int e^{-st} \cdot t dt &= \int -\frac{1}{s} (e^{-st})' \cdot t dt \\ &= -\frac{t}{s} e^{-st} + \int \frac{1}{s} e^{-st} dt \\ &= -\frac{t}{s} e^{-st} - \frac{1}{s^2} e^{-st}\end{aligned}$$

3



$$f(t) = \begin{cases} 1-t, & 0 \leq t < 1 \\ 0, & t \geq 1 \end{cases}$$

$$\begin{aligned} \mathcal{L}\{f(t)\} &= \int_0^1 (1-t) e^{-st} dt = \int_0^1 e^{-st} dt - \int_0^1 t e^{-st} dt \\ &= -\frac{1}{s} e^{-st} \Big|_{t=0}^1 + \left(\frac{t}{s} e^{-st} + \frac{1}{s^2} e^{-st} \right) \Big|_{t=0}^1 \\ &= -\frac{1}{s} e^{-s} + \frac{1}{s} + \frac{1}{s} e^{-s} + \frac{1}{s^2} e^{-s} - \frac{1}{s^2} \end{aligned}$$

$$\mathcal{L}\{f(t)\} = \frac{1}{s} - \frac{1}{s^2} + \frac{1}{s^2} e^{-s} ; s > 0$$

4

$$f(t) = e^{t+7}$$

$$\begin{aligned} \mathcal{L}\{e^{t+7}\} &= \int_0^{\infty} e^{-st} e^{t+7} dt = e^7 \int_0^{\infty} e^{(1-s)t} dt \\ &= e^7 \cdot \frac{1}{1-s} e^{(1-s)t} \Big|_{t=0}^{\infty} \quad \text{converges iff } 1-s < 0 ; s > 1 \end{aligned}$$

$$= e^7 \frac{-1}{1-s} \Rightarrow \mathcal{L}\{e^{t+7}\} = \frac{e^7}{s-1} ; s > 1$$

5

$$f(t) = t e^{4t}$$

$$\begin{aligned} \mathcal{L}\{t e^{4t}\} &= \int_0^{\infty} e^{-st} \cdot t e^{4t} dt = \int_0^{\infty} t e^{(4-s)t} dt \\ &= \int_0^{\infty} t \cdot \frac{1}{4-s} (e^{(4-s)t})' dt \\ &= \frac{t}{4-s} e^{(4-s)t} \Big|_{t=0}^{\infty} - \int_0^{\infty} \frac{1}{4-s} e^{(4-s)t} dt \\ &= 0 - \frac{1}{(4-s)^2} e^{(4-s)t} \Big|_0^{\infty} = \frac{1}{(4-s)^2} \end{aligned}$$

Need $(4-s) < 0$
 $s > 4$

$$\mathcal{L}\{t e^{4t}\} = \frac{1}{(4-s)^2} ; s > 4$$

⑥ $f(t) = t \cos t$

$$\begin{aligned} \mathcal{L}\{t \cos t\} &= \int_0^{\infty} e^{-st} \cdot t \cos t \, dt \\ &= \int_0^{\infty} -\frac{1}{s} (e^{-st})' \cdot t \cos t \, dt \\ &= -\frac{1}{s} e^{-st} \cdot t \cos t \Big|_{t=0}^{\infty} + \frac{1}{s} \int_0^{\infty} e^{-st} (\cos t - t \sin t) \, dt \\ &= \frac{1}{s} \int_0^{\infty} e^{-st} \cos t \, dt - \frac{1}{s} \int_0^{\infty} e^{-st} \cdot t \sin t \, dt \\ &= \frac{1}{s} \mathcal{L}\{\cos t\} - \frac{1}{s} \mathcal{L}\{t \sin t\} \end{aligned}$$

$$\boxed{\mathcal{L}\{t \cos t\} = \frac{1}{s} \mathcal{L}\{\cos t\} - \frac{1}{s} \mathcal{L}\{t \sin t\}} \quad (1)$$

$$\begin{aligned} \mathcal{L}\{\cos t\} &= \int_0^{\infty} e^{-st} \cos t \, dt \\ &= \int_0^{\infty} -\frac{1}{s} (e^{-st})' \cos t \, dt \\ &= -\frac{1}{s} e^{-st} \cos t \Big|_0^{\infty} - \frac{1}{s} \int_0^{\infty} e^{-st} \sin t \, dt \quad (s > 0) \\ &= \frac{1}{s} + \frac{1}{s^2} \int_0^{\infty} (e^{-st})' \sin t \, dt \\ &= \frac{1}{s} + \frac{1}{s^2} e^{-st} \sin t \Big|_{t=0}^{\infty} - \frac{1}{s^2} \int_0^{\infty} e^{-st} \cos t \, dt \\ &\quad \underbrace{\hspace{10em}}_{\mathcal{L}\{\cos t\}} \end{aligned}$$

$$\Rightarrow \mathcal{L}\{\cos t\} = \frac{1}{s} - \frac{1}{s^2} \mathcal{L}\{\cos t\} \quad ; s > 0$$

$$\Rightarrow \frac{s^2+1}{s^2} \mathcal{L}\{\cos t\} = \frac{1}{s} \Rightarrow \boxed{\mathcal{L}\{\cos t\} = \frac{s}{s^2+1}} \quad (2)$$

$$\begin{aligned}
 \mathcal{L}\{t \cos t\} &= \int_0^{\infty} e^{-st} t \cos t \, dt \\
 &= \int_0^{\infty} \frac{-1}{s} (e^{-st})' t \cos t \, dt \\
 &= \underbrace{\frac{-1}{s} e^{-st} t \cos t}_{0} \Big|_{t=0}^{\infty} + \int_0^{\infty} \frac{1}{s} e^{-st} (\cos t + t \sin t) \, dt
 \end{aligned}$$

$$\Rightarrow \boxed{\mathcal{L}\{t \cos t\} = \frac{1}{s} \mathcal{L}\{\cos t\} + \frac{1}{s} \mathcal{L}\{t \sin t\}} \quad (3)$$

$$\begin{aligned}
 \mathcal{L}\{\cos t\} &= \int_0^{\infty} e^{-st} \cos t \, dt = \int_0^{\infty} -\frac{1}{s} (e^{-st})' \cos t \, dt \\
 &= -\frac{1}{s} e^{-st} \cos t \Big|_0^{\infty} + \int_0^{\infty} \frac{1}{s} e^{-st} \sin t \, dt \\
 &= \frac{1}{s} \mathcal{L}\{\sin t\} = \frac{1}{s^2+1} \quad (\text{from (2)})
 \end{aligned}$$

$$\Rightarrow (3): \boxed{\mathcal{L}\{t \cos t\} = \frac{1}{s} \frac{1}{s^2+1} + \frac{1}{s} \mathcal{L}\{t \cos t\}} \quad (4)$$

$\Rightarrow (1), (2) \& (4):$

$$\begin{aligned}
 \mathcal{L}\{t \cos t\} &= \frac{1}{s} \cdot \frac{1}{s^2+1} - \frac{1}{s} \left(\frac{1}{s} \frac{1}{s^2+1} + \frac{1}{s} \mathcal{L}\{t \cos t\} \right) \\
 &= \frac{1}{s^2+1} - \frac{1}{s^2(s^2+1)} - \frac{1}{s^2} \mathcal{L}\{t \cos t\}
 \end{aligned}$$

$$\Rightarrow \frac{s^2+1}{s^2} \mathcal{L}\{t \cos t\} = \frac{s^2-1}{s^2(s^2+1)} \Rightarrow \boxed{\mathcal{L}\{t \cos t\} = \frac{s^2-1}{(s^2+1)^2}, s > 0}$$

⑦ $f(t) = t \sin t$

$$\text{From (4): } \mathcal{L}\{t \sin t\} = \frac{1}{s(s^2+1)} + \frac{s^2-1}{s(s^2+1)^2} = \frac{s^2+1+s^2-1}{s(s^2+1)^2}$$

$$\boxed{\mathcal{L}\{t \sin t\} = \frac{2s}{(s^2+1)^2}; s > 0}$$

⑧ $f(t) = e^{-t} \cos t$

$$\mathcal{L}\{e^{-t} \cos t\} = \int_0^{\infty} e^{-st} \cdot e^{-t} \cos t \, dt$$

$$= \int_0^{\infty} e^{-(s+1)t} \cos t \, dt$$

$$= \int_0^{\infty} -\frac{1}{s+1} e^{-(s+1)t} \cos t \, dt$$

$$= -\frac{1}{s+1} e^{-(s+1)t} \cos t \Big|_{t=0}^{\infty} + \int_0^{\infty} \frac{1}{s+1} e^{-(s+1)t} \sin t \, dt$$

0 if $s+1 > 0$; $s > -1$

$$= -\frac{1}{(s+1)^2} \int_0^{\infty} (e^{-(s+1)t})' \cos t \, dt$$

$$= -\frac{1}{(s+1)^2} e^{-(s+1)t} \cos t \Big|_0^{\infty} - \frac{1}{(s+1)^2} \int_0^{\infty} e^{-(s+1)t} \sin t \, dt$$

$\frac{1}{(s+1)^2}$ if $s > -1$

$$\Rightarrow \mathcal{L}\{e^{-t} \cos t\} = \frac{1}{(s+1)^2} - \frac{1}{(s+1)^2} \mathcal{L}\{e^{-t} \cos t\}$$

$$\Rightarrow \frac{(s+1)^2 + 1}{(s+1)^2} \mathcal{L}\{e^{-t} \cos t\} = \frac{1}{(s+1)^2} \Rightarrow \mathcal{L}\{e^{-t} \cos t\} = \frac{1}{s^2 + 2s + 2}; \quad s > -1$$