Math 2552 - Differential Equations **Homework 10**
Sections F1 – F4; L1 – L4
Georgia Institute of Technology, Fall 2015

Variation of Parameters
Cauchy-Euler Equation

Find the general solution for the differential equations below. For each one, give AN interval where your solution is valid. (Just one that works, is all I'm asking.)

1. $y'' + 3y' + 2y = \dfrac{1}{1+e^x}$.

2. $x^2y'' - 3xy' - 2y = 0$.

3. $xy'' + y' = 0$.

4. $x^2y'' - xy' + y = \ln x$.

5. $y'' + 3y' + 2y = \sin(e^x)$.

6. $y'' + 2y' + y = e^{-x} \ln x$.

7. $3x^2y'' + 6xy' + y = 0$.

8. $x^2y'' + 5xy' + 4y = 0$.

9. $x^2y'' - xy' + y = 4x \ln x$.

10. $x^2y'' - xy' + 2y = 0$.

11. $y'' - 2y' + y = \dfrac{e^x}{x}$.

12. $y'' + 4y = \sin^2(2x)$.

13. $x^2y'' - xy' = x^3e^x$.

14. $25x^2y'' + 25xy' + y = 0$.

15. $y''' - 3y'' + 3y' - y = \dfrac{e^x}{x}$.

16. $y''' - 3y'' + 2y' = \dfrac{e^{3x}}{1+e^x}$.

17. $x^2y'' + xy' + y = \sec(\ln x)$.

4.7 Variation of Parameters

$$\boxed{y'' + P(x)y' + Q(x)y = g(x)} \rightarrow \text{Standard Form}$$

1. Find the complementary solution $y_c = C_1 \underbrace{y_1 + C_2 y_2}$ linearly independent (fundamental set)

2. Compute the determinants:

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \quad \text{(the Wronskian, never 0)}$$

$$W_1 = \begin{vmatrix} 0 & y_2 \\ g(x) & y_2' \end{vmatrix}$$

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & g(x) \end{vmatrix}$$

These come from Cramer's Rule

3. Find the functions $u_1(x), u_2(x)$ given by: $\boxed{u_1' = \frac{W_1}{W} \; ; \; u_2' = \frac{W_2}{W}}$

4. Particular Solution: $\boxed{y_p = u_1 y_1 + u_2 y_2}$

5. General Solution: $y = c_1 y_1 + c_2 y_2 + u_1 y_1 + u_2 y_2$

General Higher Order: $\boxed{y^{(n)} + P_{n-1}(x) y^{(n-1)} + \ldots + P_1(x) y' + P_0(x) y = g(x)}$

1. Find complementary solution: $y_c = C_1 y_1 + C_2 y_2 + \ldots + C_n y_n$

2. Compute the Wronskian of the fundamental set:

$$W = \begin{vmatrix} y_1 & y_2 & \ldots & y_n \\ y_1' & y_2' & \ldots & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \ldots & y_n^{(n-1)} \end{vmatrix}$$

And for every $k \in \{1, 2, \ldots, n\}$, compute the determinant W_k obtained by replacing the k^{th} column of W by: $\begin{bmatrix} 0 \\ \vdots \\ 0 \\ g(x) \end{bmatrix}$

3. Find the functions $u_1(x), \ldots, u_n(x)$ given by: $\boxed{u_k' = \frac{W_k}{W}}$ for every $k \in \{1, \ldots, n\}$

4. Particular Solution: $\boxed{y_p = u_1 y_1 + u_2 y_2 + \ldots + u_n y_n}$

5. General Solution: $y = y_c + y_p$.

Example : $y''' - 2y'' - y' + 2y = e^{3x}$

(1) Complementary Solution : $m^3 - 2m^2 - m + 2 = 0$

$$(m^2-1)(m-2)=0 \Rightarrow m \in \{-1, 1, 2\}$$

$$y_c = c_1 e^{-x} + c_2 e^{x} + c_3 e^{2x}$$

(2) Determinants :

$$W = \begin{vmatrix} e^{-x} & e^{x} & e^{2x} \\ -e^{-x} & e^{x} & 2e^{2x} \\ e^{-x} & e^{x} & 4e^{2x} \end{vmatrix} \underset{\substack{R_1+R_2 \\ R_3-R_1}}{=} \begin{vmatrix} e^{-x} & e^{x} & e^{2x} \\ 0 & 2e^{x} & 3e^{2x} \\ 0 & 0 & 3e^{2x} \end{vmatrix} = e^{-x} \begin{vmatrix} 2e^{x} & 3e^{2x} \\ 0 & 3e^{2x} \end{vmatrix} = 6e^{2x}$$

$$W_1 = \begin{vmatrix} 0 & e^{x} & e^{2x} \\ 0 & e^{x} & 2e^{2x} \\ e^{3x} & e^{x} & 4e^{2x} \end{vmatrix} = e^{3x} \begin{vmatrix} e^{x} & e^{2x} \\ e^{x} & 2e^{2x} \end{vmatrix} = e^{6x}$$

$$W_2 = \begin{vmatrix} e^{-x} & 0 & e^{2x} \\ -e^{-x} & 0 & 2e^{2x} \\ e^{-x} & e^{3x} & 4e^{2x} \end{vmatrix} = -e^{3x} \begin{vmatrix} e^{-x} & e^{2x} \\ -e^{-x} & 2e^{2x} \end{vmatrix} = -3e^{4x}$$

position a_{32} $(-1)^{3+2} = (-1)$

$$W_3 = \begin{vmatrix} e^{-x} & e^{x} & 0 \\ -e^{-x} & e^{x} & 0 \\ e^{-x} & e^{x} & e^{3x} \end{vmatrix} = e^{3x} \begin{vmatrix} e^{-x} & e^{x} \\ -e^{-x} & e^{x} \end{vmatrix} = 2e^{3x}$$

(3) $u_1' = \dfrac{e^{6x}}{6e^{2x}} = \dfrac{1}{6}e^{4x} \Rightarrow \boxed{u_1 = \dfrac{1}{24}e^{4x}}$

$u_2' = \dfrac{-3e^{4x}}{6e^{2x}} = -\dfrac{1}{2}e^{2x} \Rightarrow \boxed{u_2 = -\dfrac{1}{4}e^{2x}}$

$u_3' = \dfrac{2e^{3x}}{6e^{2x}} = \dfrac{1}{3}e^{x} \Rightarrow \boxed{u_3 = \dfrac{1}{3}e^{x}}$

(4) $y_p = \left(\dfrac{1}{24}e^{4x}\right)e^{-x} + \left(-\dfrac{1}{4}e^{2x}\right)e^{x} + \left(\dfrac{1}{3}e^{x}\right)e^{2x} = \boxed{\dfrac{1}{8}e^{3x}}$ $\quad y_p = \dfrac{1}{8}e^{3x}$

(5) $\boxed{y = c_1 e^{-x} + c_2 e^{x} + c_3 e^{2x} + \dfrac{1}{8}e^{3x}}$

(4.3) Cauchy-Euler Equation

$$\boxed{ax^2y'' + bxy' + cy = 0}$$

Solve on $(0, \infty)$

- **Characteristic Equation:**

$$am^2 + (b-a)m + c = 0$$

- **3 Cases:**

(1) 2 distinct real roots m_1, m_2 : $\boxed{y = c_1 x^{m_1} + c_2 x^{m_2}}$

(2) 1 repeated real root m_1 : $\boxed{y = c_1 x^{m_1} + c_2 x^{m_1} \ln x}$

(3) 2 complex roots $\alpha \pm i\beta$: $\boxed{y = x^{\alpha}\left(c_1 \cos(\beta \ln x) + c_2 \sin(\beta \ln x)\right)}$

Why? Solve $\boxed{ax^2y'' + bxy' + cy = 0}$ on $(0, \infty)$.

Substitution: $\boxed{x = e^t, \text{ or } \ln x = t}$

$$\Rightarrow \frac{dy}{dx} = \frac{dy}{dt}\overbrace{\frac{dt}{dx}}^{1/x} = \frac{1}{x}\frac{dy}{dt}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dt}\right)$$

$$= -\frac{1}{x^2}\frac{dy}{dt} + \frac{d}{dx}\frac{dy}{dt}\cdot\frac{1}{x}$$

$$= -\frac{1}{x^2}\frac{dy}{dt} + \frac{d^2y}{dt^2}\cdot\frac{dt}{dx}\cdot\frac{1}{x}$$

$$= \frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$$

$$ax^2 \cdot \frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) + bx\cdot\left(\frac{1}{x}\frac{dy}{dt}\right) + cy = 0$$

$$a\frac{d^2y}{dt^2} - a\frac{dy}{dt} + b\cdot\frac{dy}{dt} + cy = 0$$

$$\boxed{a\frac{d^2y}{dt^2} + (b-a)\frac{dy}{dt} + cy = 0}$$

Constant Coeff. w/ Char. eqn.: $\boxed{am^2 + (b-a)m + c = 0}$

- 2 distinct real roots $m_1, m_2 \Rightarrow y = c_1 e^{m_1 t} + c_2 e^{m_2 t} = c_1 x^{m_1} + c_2 x^{m_2}$ $(t = \ln x)$
- 1 repeated real root $m_1 \Rightarrow y = c_1 e^{m_1 t} + c_2 t e^{m_1 t} = c_1 x^{m_1} + c_2 x^{m_1} \ln x.$
- 2 complex roots $\alpha \pm i\beta \Rightarrow y = e^{\alpha t}\left(c_1 \cos(\beta t) + c_2 \sin(\beta t)\right)$

$$= x^{\alpha}\left(c_1 \cos(\beta \ln x) + c_2 \sin(\beta \ln x)\right).$$