Name: Solutions

December 10^{th}, 2015.
Math 2552; Sections L1 – L4.
Georgia Institute of Technology
FINAL EXAM

I commit to uphold the ideals of honor and integrity by refusing to betray the trust bestowed upon me as a member of the Georgia Tech community. By signing my name below I pledge that I have neither given nor received help on this exam.

Pledged: ____________________

Problem	Possible Score	Earned Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
11	10	
12	10	
13	10	
14	10	
Bonus	10	
Total	140	

Remember that you must SHOW YOUR WORK to receive credit!

Good luck!

1. [10 points] Find the inverse Laplace transforms of the following functions:

a). [3 points] $\mathscr{L}^{-1}\left\{\frac{3s-1}{s^2+16}\right\} = 3\mathscr{L}^{-1}\left\{\frac{s}{s^2+16}\right\} - \mathscr{L}^{-1}\left\{\frac{1}{s^2+16}\right\}$

$$= 3\cos(4t) - \frac{1}{4}\sin(4t)$$

b). [4 points] $\mathscr{L}^{-1}\left\{\frac{s+3}{s(s+2)}\right\}. = \frac{3}{2}\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{1}{2}\mathscr{L}^{-1}\left\{\frac{1}{s+2}\right\} = \frac{3}{2} - \frac{1}{2}e^{-2t}.$

$$\frac{s+3}{s(s+2)} = \frac{A}{s} + \frac{B}{s+2}$$

$$A = \frac{3}{2}\ ;\ B = -\frac{1}{2}$$

c). [3 points] $\mathscr{L}^{-1}\left\{\frac{2s}{s^2+1}e^{-4s}\right\} = \mathscr{L}^{-1}\left\{\frac{2s}{s^2+1}\right\}\Big|_{t\mapsto t-4} u_4(t)$

$$= 2\cos(t-4)\,u_4(t)$$

2. [10 points] Suppose that

$$\mathbf{v_1}(t) = \begin{pmatrix} e^t \\ e^t - e^{-t} \end{pmatrix} \quad \text{and} \quad \mathbf{v_2}(t) = \begin{pmatrix} 0 \\ e^{-t} \end{pmatrix}$$

are solutions to the linear system $\mathbf{x}' = A\mathbf{x}$, where A is a real 2×2 matrix. Find a particular solution to the linear system:

$$\mathbf{x}' = A\mathbf{x} + \begin{pmatrix} e^t \\ e^{-t} \end{pmatrix}.$$

[Hint:] Compute $\mathbf{v_1}(0)$ and $\mathbf{v_2}(0)$.

4pts.

$$\vec{v}_1(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} ; \vec{v}_2(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow \boxed{e^{tA} = \begin{pmatrix} e^t & 0 \\ e^t - e^{-t} & e^{-t} \end{pmatrix}} \; !$$

2pts.

$$\Rightarrow e^{-tA}\vec{g}(t) = \begin{pmatrix} e^{-t} & 0 \\ e^{-t} - e^{t} & e^t \end{pmatrix} \begin{pmatrix} e^t \\ e^{-t} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 - e^{2t} + 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 - e^{2t} \end{pmatrix}$$

2pts.

$$\Rightarrow \int e^{-tA}\vec{g}(t)\,dt = \begin{pmatrix} t \\ 2t - \frac{1}{2}e^{2t} \end{pmatrix}$$

$$\Rightarrow \vec{X}_p = e^{tA}\int e^{-tA}\vec{g}(t)\,dt = \begin{pmatrix} e^t & 0 \\ e^t - e^{-t} & e^{-t} \end{pmatrix} \begin{pmatrix} t \\ 2t - \frac{1}{2}e^{2t} \end{pmatrix}$$

$$= \begin{pmatrix} te^t \\ te^t - te^{-t} + 2te^{-t} - \frac{1}{2}e^t \end{pmatrix} = \begin{pmatrix} te^t \\ te^t - \frac{1}{2}e^t + te^{-t} \end{pmatrix}$$

2pts.

$$\boxed{\vec{X}_p = \begin{pmatrix} te^t \\ te^t - \frac{1}{2}e^t + te^{-t} \end{pmatrix}}$$

3. [10 points] Consider the autonomous equation:

$$y' = (y^2 - 1)(3 - y).$$

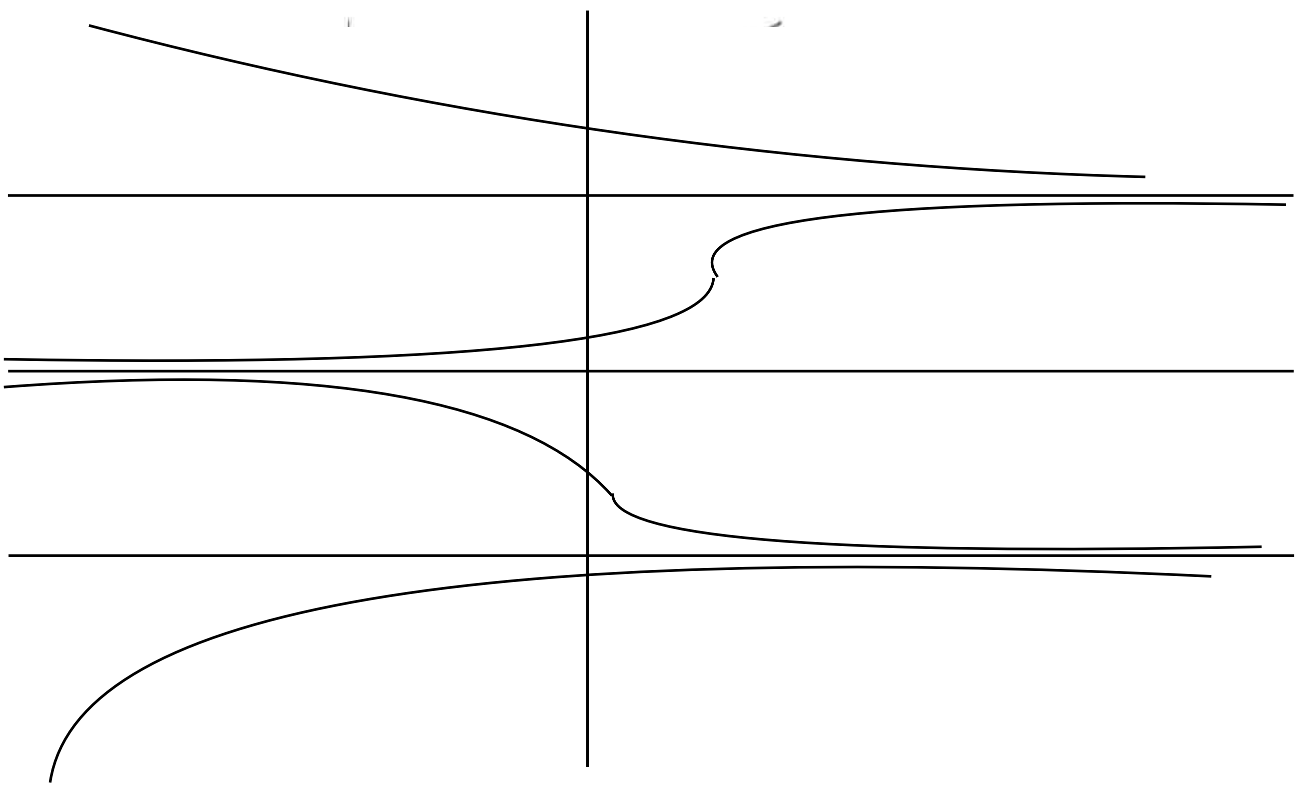
a). [3 points] What are the equilibrium solutions of this equation?

$y = 1$; $y = -1$; $y = 3$

b). [4 points] Draw the phase portrait of this equation.

y		-1		1		3	
y^2-1	+	0	−	0	+		+
$3-y$	+		+		+	0	−
y'	+	0	−	0	+	0	−

c). [3 points] Draw a picture of what the solution curves for this equation might look like.



4. [10 points] Find a particular solution to the equation:

$$y'' - 3y' + 2y = \frac{1}{1+e^x},$$

given that

$$y_1(x) = e^x \quad \text{and} \quad y_2(x) = e^{2x}$$

are two solutions of the associated homogeneous equation $y'' + 3y' + 2y = 0$.

$$W = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix} = 2e^{3x} - e^{3x} = \boxed{e^{3x}}$$

$$W_1 = \begin{vmatrix} 0 & e^{2x} \\ \frac{1}{1+e^x} & 2e^{2x} \end{vmatrix} = \boxed{-\frac{e^{2x}}{1+e^x}} \Rightarrow \boxed{u_1' = -\frac{1}{e^x(1+e^x)}}$$

$$W_2 = \begin{vmatrix} e^x & 0 \\ e^x & \frac{1}{1+e^x} \end{vmatrix} = \boxed{\frac{e^x}{1+e^x}} \Rightarrow \boxed{u_2' = \frac{1}{e^{2x}(1+e^x)}}$$

$$\Rightarrow u_1 = -\int \frac{1}{e^x(1+e^x)}dx = -\int \frac{(1+e^x)-e^x}{e^x(1+e^x)}dx = -\int\left(\frac{1}{e^x} - \frac{1}{1+e^x}\right)dx$$

$$= -\int e^{-x}dx + \int \frac{(1+e^x)-e^x}{1+e^x}dx = e^{-x} + \int\left(1 - \frac{e^x}{1+e^x}\right)dx$$

$$= \boxed{e^{-x} + x - \ln(1+e^x)}$$

$$\Rightarrow u_2 = \int \frac{1}{e^{2x}(1+e^x)}dx = \int \frac{(1+e^x)-e^x}{e^{2x}(1+e^x)}dx = \int\left(\frac{1}{e^{2x}} - \frac{1}{e^x(1+e^x)}\right)dx$$

$$= \int e^{-2x}dx - \int \frac{1}{e^x(1+e^x)}dx$$

$$= \boxed{-\frac{1}{2}e^{-2x} + e^{-x} + x - \ln(1+e^x)}$$

can be absorbed back into y_c)

$$\Rightarrow y_p = 1 + xe^x - e^x\ln(1+e^x) - \frac{1}{2} + e^x + xe^{2x} - e^{2x}\ln(1+e^x)$$

$$\Rightarrow y_p = \boxed{\frac{1}{2} + xe^x + xe^{2x} - e^x\ln(1+e^x) - e^{2x}\ln(1+e^x)} = \boxed{\frac{1}{2} + (e^x + e^{2x})(x - \ln(1+e^x))}$$

5. [10 points]

a). [3 points] Find: $\mathscr{L}^{-1}\left\{\frac{1}{s^2+6s+13}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{(s+3)^2+4}\right\}$

$$= \mathscr{L}^{-1}\left\{\frac{1}{s^2+4}\Big|_{s\to s+3}\right\} = e^{-3t}\frac{1}{2}\sin(2t).$$

Completing the square : (2pts)
Answer : (1pt.)

b). [4 points] Find: $\mathscr{L}^{-1}\left\{\frac{5s+35}{s^2+6s+13}\right\} = \mathscr{L}^{-1}\left\{\frac{5s+35}{(s+3)^2+4}\right\}$

$$= \mathscr{L}^{-1}\left\{\frac{5(s+3)}{(s+3)^2+4}\right\} + \mathscr{L}^{-1}\left\{\frac{20}{(s+3)^2+4}\right\}$$

$$= 5\mathscr{L}^{-1}\left\{\frac{s}{s^2+4}\Big|_{s\to s+3}\right\} + 20\mathscr{L}^{-1}\left\{\frac{1}{s^2+4}\Big|_{s\to s+3}\right\}$$

$$= 5e^{-3t}\cos(2t) + 10e^{-3t}\sin(2t)$$

Correct Separation: (3pts.)
Answer: (1pt.)

c). [3 points] Solve the initial value problem:

$$y'' + 6y' + 13y = \delta(t-1); \quad y(0) = 5, y'(0) = 5.$$

Take Laplace of both sides:

$$s^2Y(s) - 5s - 5 + 6sY(s) - 30 + 13Y(s) = e^{-s} \quad \text{(1pt.)}$$

$$(s^2+6s+13)Y(s) = 5s+35+e^{-s}$$

$$Y(s) = \frac{5s+35}{s^2+6s+13} + \frac{e^{-s}}{s^2+6s+13} \quad \text{(1pt.)}$$

(1pt.) applying Translation Thm. to part (a).

$$\Rightarrow y(t) = 5e^{-3t}\cos(2t) + 10e^{-3t}\sin(2t) + e^{-3(t-1)}\frac{1}{2}\sin(2t-2)\,u_1(t).$$

6. [10 points] Find a solution to the equation:

$$y(t) = e^t + \underbrace{\int_0^t \sin(t-\tau)y(\tau)\,d\tau}_{\sin(t) * y(t)}.$$

(2pts.) - recognize convolution

$$Y(s) = \frac{1}{s-1} + \frac{1}{s^2+1} Y(s)$$ (1pt.) - take Laplace of both sides

$$\frac{s^2}{s^2+1} Y(s) = \frac{1}{s-1}$$

$$Y(s) = \frac{s^2+1}{s^2(s-1)}$$ (1pt.) - obtain $Y(s)$

Partial Fraction Decomposition:

$$\frac{s^2+1}{s^2(s-1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-1}$$

$B=-1$; $C = 2$

(4pts.) (1 pt.) Setup; (1pt.) × each coefficient

$$s^2 + \cancel{1} = As(s-1) - s + \cancel{1} + 2s^2$$

$$s^2 = As^2 - As - s + 2s^2$$

$$s^2 = (A+2)s^2 - (A+1)s$$

$A = -1$

$$\Rightarrow y(t) = -\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}$$

$$\boxed{y(t) = -1 - t + 2e^t}$$ (2pts.) - Solution

[1 pt. bonus:] What is this type of equation called? Volterra Integral Equation

7. [10 points] Solve the differential equation:

$$y^{(4)} - 7y'' - 18y = 0.$$

Characteristic Equation: $m^4 - 7m^2 - 18 = 0$ (2pts.)

$u = m^2$ $\quad u^2 - 7u - 18 = 0$

$(u-9)(u+2) = 0$

$u = 9 \qquad u = -2$

$m^2 = 9 \qquad m^2 = -2$

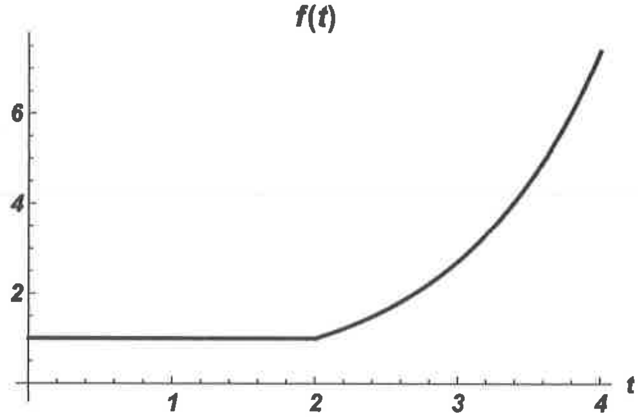
$m = \pm 3 \qquad m = \pm i\sqrt{2}$ (4pts.)

$$\boxed{y = c_1 e^{3x} + c_2 e^{-3x} + c_3 \cos(\sqrt{2}x) + c_4 \sin(\sqrt{2}x)}$$ (4pts.)

8. [10 points] The picture to the right is the graph of the function $f(t)$ obtained by:

- ☐ f is the constant function $y = 1$ from $t = 0$ to $t = 2$;
- ☐ From $t = 2$ to $t = 4$, it is the graph of $y = e^t$ shifted two units to the right.
- ☐ On $(4, \infty)$, f is the 0 function.

Compute the Laplace transform $\mathscr{L}\{f(t)\}$ of this function.



5 pts.

$$f(t) = \underbrace{(1 - u_2(t))}_{1.5 \text{ pts.}} + \underbrace{e^{t-2}}_{2 \text{ pts.}} \underbrace{(u_2(t) - u_4(t))}_{1.5 \text{ pts.}}$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{1\} - \mathcal{L}\{u_2(t)\} + \mathcal{L}\{e^{t-2}u_2(t)\} - \mathcal{L}\{e^{t-2}u_4(t)\}$$

$$= \frac{1}{s} - \frac{e^{-2s}}{s} + e^{-2s}\mathcal{L}\{e^t\} - \mathcal{L}\{e^{(t-4)+2}u_4(t)\}$$

$$= \boxed{\underbrace{\frac{1}{s} - \frac{e^{-2s}}{s}}_{(1 \text{ pt.})} + \underbrace{\frac{e^{-2s}}{s-1}}_{(2 \text{ pts.})} - \underbrace{e^2 \frac{e^{-4s}}{s-1}}_{(2 \text{ pts.})}}$$

9. [10 points] Solve the differential equation:

$$y' = -2xy^2,$$

subject to the following initial conditions:

a). $y(0) = \frac{1}{3}$.

If $y \neq 0$: $\quad -\frac{1}{y^2}dy = 2x\,dx$

$$\frac{1}{y} = x^2 + C \Rightarrow \boxed{y = \frac{1}{x^2+C}} \quad \text{or} \quad \boxed{y = 0}$$

Is $y=0$ a solution? Yes.

$$\left.\begin{array}{l} y(0) = \frac{1}{C} \\ y(0) = \frac{1}{3} \end{array}\right\} \Rightarrow C = 3 \Rightarrow \boxed{y = \frac{1}{x^2+3}}$$

b). $y(0) = 0$.

$$y(x) = \frac{1}{x^2+C} \Rightarrow y(0) = \frac{1}{C} \text{ can never be } 0$$

$$\Rightarrow \boxed{y = 0} \text{ only solution to this IVP}$$

Grading Rubric:
- Separation: 1pt.
- Treating $y=0$ separately: 1pt.
- Integrals: 2pts.
- Solution $y = \frac{1}{x^2+C}$: 1pt.
- Sol. to IVP (a): 2pts.
- Sol. to IVP (b): 3pts.

10. [10 points] Given that

$$\mathbf{x_1} = e^t \begin{pmatrix} -1 \\ 3 \end{pmatrix} \quad \text{and} \quad \mathbf{x_2} = te^t \begin{pmatrix} -1 \\ 3 \end{pmatrix} + e^t \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

are solutions to a homogeneous linear system $\mathbf{x}' = A\mathbf{x}$ for a real 2×2 matrix A, solve the initial value problem:

$$\mathbf{x}' = A\mathbf{x}; \ \mathbf{x}(0) = \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

$$\vec{x} = c_1 e^t \begin{pmatrix} -1 \\ 3 \end{pmatrix} + c_2 te^t \begin{pmatrix} -1 \\ 3 \end{pmatrix} + c_2 e^t \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$ 2pts.

$$\Rightarrow \vec{x}(0) = c_1 \begin{pmatrix} -1 \\ 3 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -c_1 \\ 3c_1 - c_2 \end{pmatrix}$$ 2pts.

$$\Rightarrow \begin{cases} -c_1 = 3 \Rightarrow c_1 = -3 \\ 3c_1 - c_2 = 1 \Rightarrow -9 - c_2 = 1 \Rightarrow c_2 = -10 \end{cases}$$ 2pts. 2pts.

$$\Rightarrow \boxed{\vec{x} = -3e^t \begin{pmatrix} -1 \\ 3 \end{pmatrix} - 10te^t \begin{pmatrix} -1 \\ 3 \end{pmatrix} - 10e^t \begin{pmatrix} 0 \\ -1 \end{pmatrix}}$$

or $$\boxed{\vec{x} = e^t \begin{pmatrix} 3 \\ 1 \end{pmatrix} - 10te^t \begin{pmatrix} -1 \\ 3 \end{pmatrix}}$$ 2pts.

11. [10 points] Solve the linear system:

$$\mathbf{x}' = \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix} \mathbf{x}.$$

Eigenvalues:

$$\begin{vmatrix} 1-\lambda & 0 \\ 2 & 3-\lambda \end{vmatrix} = (\lambda-3)(\lambda-1) \qquad \lambda_1 = 1 \quad \lambda_2 = 3$$

2pts. 2pts.

Eigenvectors:

$$\lambda_1 = 1: \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 2 & 2 & 0 \end{array}\right) \Rightarrow v_1 = -v_2 \Rightarrow \vec{v_1} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$ 2pts.

$$\lambda_2 = 3: \left(\begin{array}{cc|c} -2 & 0 & 0 \\ 2 & 0 & 0 \end{array}\right) \Rightarrow v_1 = 0 \Rightarrow \vec{v_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$ 2pts.

Solution:

$$\boxed{\vec{x} = c_1 e^{t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$$ 2pts.

12. [10 points] Find the general solution to the differential equation:

$$y' - \frac{1}{x}y = 4x^2\frac{1}{y}\cos x.$$

1pt.

Bernoulli with $\alpha = -1$ $\Rightarrow 1-\alpha = 2 \Rightarrow$ $u = y^2$ 2pts.

$u' - \frac{2}{x}u = 8x^2\cos x$ 1pt.

Linear: $\mu(x) = \frac{1}{x^2}$

2pts.

$\Rightarrow \frac{d}{dx}\left(\frac{1}{x^2}u\right) = 8\cos x$

$\Rightarrow \frac{1}{x^2}u = 8\sin x + C$ 1pt.

$\Rightarrow u = 8x^2\sin x + Cx^2$ 1pt.

$\Rightarrow \boxed{y^2 = 8x^2\sin x + Cx^2}$ 2pts.

13. [10 points] Solve the differential equation:

$$x^2y'' - xy' + 5y = 0,\ x > 0.$$

Cauchy-Euler with $a=1$; $b=-1$; $c=5$ (1pt.)

Char. Eqn.: $m^2-2m+5=0$ (2pts.)

$\Delta = 4-20=-16$

$m = \frac{2 \pm 4i}{2} = 1 \pm 2i$ (3pts.)

$$\boxed{y = x\left(c_1 \cos(2\ln x) + c_2 \sin(2\ln x)\right)}$$ (4pts.)

14. [10 points] Find the general solution to the differential equation:

$$\left(\frac{1}{1+y^2} + \cos x - 2xy\right) y' = y(y + \sin x)$$

(An implicit solution is fine.)

$$\left(\frac{1}{1+y^2} + \cos x - 2xy\right) dy - y(y + \sin x)\, dx = 0$$

$$y(y + \sin x)\, dx - \left(\frac{1}{1+y^2} + \cos x - 2xy\right) dy = 0$$

$$\frac{\partial M}{\partial y} = 2y + \sin x \ ; \quad \frac{\partial N}{\partial x} = \sin x + 2y \quad \Rightarrow \underline{\text{Exact}} \quad \text{(2pts.)}$$

Potential: (7pts.)

$$\frac{\partial f}{\partial x} = y^2 + y\sin x \Rightarrow f(x,y) = y^2 x - y\cos x + g(y)$$

$$\left.\begin{aligned} \Rightarrow \frac{\partial f}{\partial y} &= 2yx - \cos x + g'(y) \\ &= 2xy - \cos x - \frac{1}{1+y^2} \end{aligned}\right\} \Rightarrow$$

$$g'(y) = -\frac{1}{1+y^2} \Rightarrow g = -\arctan y$$

$$\Rightarrow \boxed{xy^2 - y\cos x - \arctan y = C} \quad \text{(1pt.)}$$

Bonus Problem: [10 points] Solve the differential equation for the function $y(x)$:

$$y' + 1 = e^{-(x+y)} \sin x.$$

Give an implicit answer.

$e^y y' + e^y = e^{-x} \sin x$

$u = e^y \Rightarrow u' = e^y y'$

$u' + u = e^{-x} \sin x$ (Linear)

$\mu(x) = e^x$

$\frac{d}{dx}(ue^x) = \sin x$

$ue^x = -\cos x + C$

$u = -e^{-x}\cos x + Ce^{-x}$

$\boxed{e^y = -e^{-x}\cos x + Ce^{-x}}$