Name: ________________________________

December 8^{th}, 2015.
Math 2552; Sections F1 – F4.
Georgia Institute of Technology
FINAL EXAM

I commit to uphold the ideals of honor and integrity by refusing to betray the trust bestowed upon me as a member of the Georgia Tech community. By signing my name below I pledge that I have neither given nor received help on this exam.

Pledged: ________________________________

Problem	Possible Score	Earned Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
11	10	
12	10	
13	10	
14	10	
Bonus	10	
Total	140	

Remember that you must SHOW YOUR WORK to receive credit!

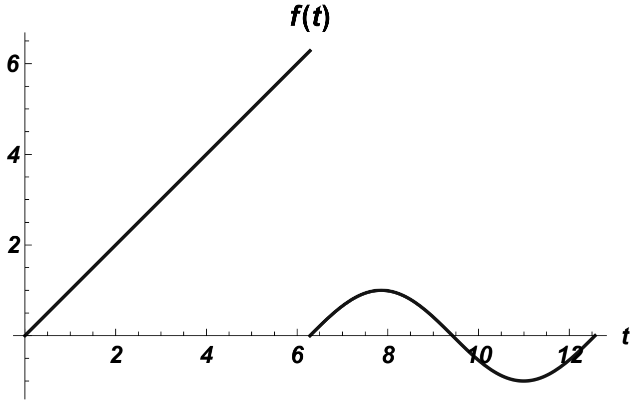
Good luck!

1. [10 points] Find the general solution to the differential equation:

$$y' - \frac{4}{x}y = 6x^2 y\sqrt{y}.$$

Give an explicit answer.

2. [10 points] The picture to the right is the graph of the function $f(t)$ obtained by:
☐ f is the linear function $y = t$ from $t = 0$ to $t = 2\pi$;
☐ From $t = 2\pi$ to $t = 4\pi$, it is the graph of $y = \sin t$.
☐ On $(4\pi, \infty)$, f is the 0 function.
Compute the Laplace transform $\mathscr{L}\{f(t)\}$ of this function.



3. [10 points] Solve the differential equation:

$$y^{(5)} - 16y' = 0.$$

4. [10 points] Find the matrix exponential e^{tA} for the matrix:

$$A = \begin{pmatrix} -1 & 1 \\ 0 & 2 \end{pmatrix}.$$

5. [10 points] Solve the differential equation:

$$y' + y^2 \cos x = 0,$$

subject to the following initial conditions:

a). $y(0) = \frac{1}{2}$.

b). $y(0) = 0$.

6. [10 points] Find a particular solution to the equation:

$$y'' - \frac{1}{x}y' + \frac{1}{x^2}y = \ln x,\ x > 0,$$

given that

$$y_1(x) = x \quad \text{and} \quad y_2(x) = x \ln x$$

are two solutions of the associated homogeneous equation $y'' - \frac{1}{x}y' + \frac{1}{x^2}y = 0$.

7. [10 points] Solve the differential equation:

$$x^2y'' - xy' + 2y = 0,\ x > 0.$$

8. [10 points] Find a solution to the equation:

$$y(t) = e^{2t} - \int_0^t (t-\tau)y(\tau)\,d\tau.$$

[1 pt. bonus:] What is this type of equation called?

9. [10 points]

a). [3 points] Find: $\mathscr{L}^{-1}\left\{\dfrac{1}{s^2-4s+13}\right\}$.

b). [4 points] Find: $\mathscr{L}^{-1}\left\{\dfrac{3s-12}{s^2-4s+13}\right\}$.

c). [3 points] Solve the initial value problem:

$$y'' - 4y' + 13y = \delta(t-3); \quad y(0) = 3, y'(0) = 0.$$

10. [10 points] Given that

$$\mathbf{x_1} = e^{2t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad \text{and} \quad \mathbf{x_2} = e^{7t}\begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

are solutions to a homogeneous linear system $\mathbf{x}' = A\mathbf{x}$ for a real 2×2 matrix A, solve the initial value problem:

$$\mathbf{x}' = A\mathbf{x}; \;\; \mathbf{x}(0) = \begin{pmatrix} 3 \\ 4 \end{pmatrix}.$$

11. [10 points] Suppose that

$$\Phi(t) = \begin{pmatrix} e^t & 2e^{2t} \\ 0 & e^{2t} \end{pmatrix}$$

is a fundamental matrix for the linear system $\mathbf{x}' = A\mathbf{x}$, where A is a real 2×2 matrix. Find a particular solution to the linear system:

$$\mathbf{x}' = A\mathbf{x} + \begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix}.$$

12. [10 points] Find the general solution to the differential equation:

$$\cos(xy) - xy\sin(xy) - x^2\sin(xy)y' = 0.$$

(An implicit solution is fine.)

13. [10 points] Consider the autonomous equation:

$$y' = (y^2 + 4)(y - 1)(y + 2).$$

a). [3 points] What are the equilibrium solutions of this equation?

b). [4 points] Draw the phase portrait of this equation.

c). [3 points] Draw a picture of what the solution curves for this equation might look like.

14. [10 points] Find the inverse Laplace transforms of the following functions:

a). [3 points] $\mathscr{L}^{-1}\left\{\dfrac{s+1}{s^2+4}\right\}$.

b). [4 points] $\mathscr{L}^{-1}\left\{\dfrac{2s-4}{(s-1)(s+2)}\right\}$.

c). [3 points] $\mathscr{L}^{-1}\left\{\dfrac{1}{s^2+9}e^{-s}\right\}$.

Bonus Problem: [10 points] Solve the differential equation for the function $y(x)$:

$$x^2y' + 2xy = x^4y^2 + 1.$$

Give an explicit answer.

Laplace transforms of some basic functions

$\mathscr{L}\{1\} = \frac{1}{s};\ s > 0$

$\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}};\ s > 0$

$\mathscr{L}\{e^{kt}\} = \frac{1}{s-k};\ s > k$

$\mathscr{L}\{\sin(kt)\} = \frac{k}{s^2+k^2};\ s > 0$

$\mathscr{L}\{\cos(kt)\} = \frac{s}{s^2+k^2};\ s > 0$

$\mathscr{L}\{\sinh(kt)\} = \frac{k}{s^2-k^2};\ s > |k|$

$\mathscr{L}\{\cosh(kt)\} = \frac{s}{s^2-k^2};\ s > |k|$

$\mathscr{L}\{u_a(t)\} = \frac{e^{-as}}{s};\ s > 0$

$\mathscr{L}\{\delta(t-t_0)\} = e^{-st_0}$

$\mathscr{L}\{\delta(t)\} = 1$

Inverse Laplace transforms of some basic functions

$\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} = 1$

$\mathscr{L}^{-1}\left\{\frac{1}{s^n}\right\} = \frac{1}{(n-1)!}t^{n-1}$

$\mathscr{L}^{-1}\left\{\frac{1}{s-k}\right\} = e^{kt}$

$\mathscr{L}^{-1}\left\{\frac{1}{s^2+k^2}\right\} = \frac{1}{k}\sin(kt)$

$\mathscr{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\} = \cos(kt)$

$\mathscr{L}^{-1}\left\{\frac{1}{s^2-k^2}\right\} = \frac{1}{k}\sinh(kt)$

$\mathscr{L}^{-1}\left\{\frac{s}{s^2-k^2}\right\} = \cosh(kt)$

$\mathscr{L}^{-1}\left\{\frac{e^{-as}}{s}\right\} = u_a(t)$

$\mathscr{L}^{-1}\left\{e^{-st_0}\right\} = \delta(t-t_0)$

$\mathscr{L}^{-1}\{1\} = \delta(t)$

Properties of the Laplace and Inverse Laplace transform

Translation Theorem I:

$\mathscr{L}\left\{e^{kt}f(t)\right\} = F(s-k) = \mathscr{L}\{f(t)\}|_{s\to s-k}$

$\mathscr{L}^{-1}\{F(s-k)\} = \mathscr{L}^{-1}\left\{F(s)|_{s\to s-k}\right\} = e^{kt}\mathscr{L}^{-1}\{F(s)\} = e^{kt}f(t)$

Translation Theorem II:

$\mathscr{L}\{f(t-a)u_a(t)\} = e^{-as}F(s) = e^{-as}\mathscr{L}\{f(t)\}$

$\mathscr{L}^{-1}\{e^{-as}F(s)\} = f(t-a)u_a(t) = \mathscr{L}^{-1}\{F(s)\}\big|_{t\to t-a}\, u_a(t)$

Derivatives of Laplace Transforms:

$\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}F(s)$

$\mathscr{L}^{-1}\left\{F^{(n)}(s)\right\} = (-1)^n t^n f(t)$

Laplace Transforms of Derivatives:

$$\begin{aligned}
\mathscr{L}\{y'\} &= sY(s) - y(0)\\
\mathscr{L}\{y''\} &= s^2Y(s) - sy(0) - y'(0)\\
&\vdots\\
\mathscr{L}\{y^{(n)}(t)\} &= s^nY(s) - s^{n-1}y(0) - s^{n-2}y'(0) - \ldots - sy^{(n-2)}(0) - y^{(n-1)}(0)
\end{aligned}$$

Convolution Theorem:

$\mathscr{L}\{f(t)\star g(t)\} = F(s)G(s) = \mathscr{L}\{f(t)\}\mathscr{L}\{g(t)\}$

$\mathscr{L}^{-1}\{F(s)G(s)\} = f(t)\star g(t)$

Dirac Delta Function:

$\int_0^\infty f(t)\delta(t-t_0)\,dt = f(t_0)$

$(f\star\delta)(t) = f(t)$

Laplace Transform of Periodic Functions:
If $f(t)$ is piecewise continuous on $[0,\infty)$, of exponential order, and periodic with period T:

$$\mathscr{L}\{f(t)\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)\,dt$$