Name: Solutions

December 8^{th}, 2015.
Math 2552; Sections F1 – F4.
Georgia Institute of Technology
FINAL EXAM

I commit to uphold the ideals of honor and integrity by refusing to betray the trust bestowed upon me as a member of the Georgia Tech community. By signing my name below I pledge that I have neither given nor received help on this exam.

Pledged: ______________________________

Problem	Possible Score	Earned Score
1	10	10
2	10	10
3	10	10
4	10	10
5	10	10
6	10	10
7	10	10
8	10	10
9	10	10
10	10	10
11	10	10
12	10	10
13	10	10
14	10	10
Bonus	10	10
Total	140	

Remember that you must SHOW YOUR WORK to receive credit!

Good luck!

1. [10 points] Find the general solution to the differential equation:

$$y' - \frac{4}{x}y = 6x^2 y\sqrt{y}.$$

1pt.

Bernoulli with $\alpha = \frac{3}{2} \Rightarrow 1-\alpha = -\frac{1}{2} \Rightarrow u = y^{-1/2}$ 2pts.

$\Rightarrow u' + \frac{2}{x}u = -3x^2$ 1pt.

Linear: $\mu(x) = x^2$ 2pts. $\Rightarrow \frac{d}{dx}(ux^2) = -3x^4$ 1pt.

$ux^2 = -\frac{3}{5}x^5 + C$

$\Rightarrow u = -\frac{3}{5}x^3 + Cx^{-2}$ 1pt.

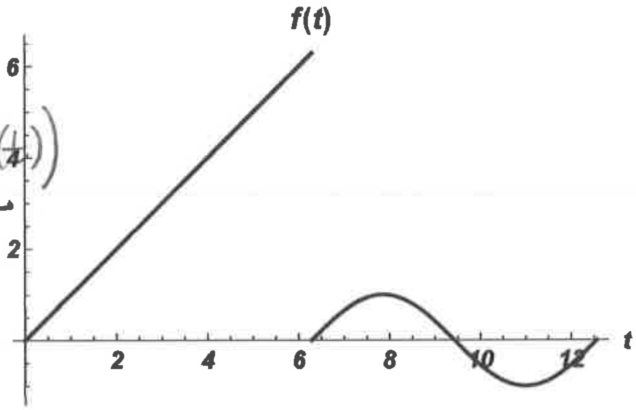
$\Rightarrow \frac{1}{\sqrt{y}} = -\frac{3}{5}x^3 + Cx^{-2}$

$\Rightarrow \boxed{y = \frac{1}{\left(-\frac{3}{5}x^3 + Cx^{-2}\right)^2}}$ 2pts.

2. [10 points] The picture to the right is the graph of the function $f(t)$ obtained by:

- ☐ f is the linear function $y = t$ from $t = 0$ to $t = 2\pi$;
- ☐ From $t = 2\pi$ to $t = 4\pi$, it is the graph of $y = \sin t$.
- ☐ On $(4\pi, \infty)$, f is the 0 function.

Compute the Laplace transform $\mathscr{L}\{f(t)\}$ of this function.



5 pts.

$$f(t) = \underbrace{t}_{(1\text{pt.})}\underbrace{\left(1 - u_{2\pi}(t)\right)}_{(1.5\text{pts.})} + \underbrace{\sin t}_{(1\text{pt.})}\underbrace{\left(u_{2\pi}(t) - u_{4\pi}(t)\right)}_{(1.5\text{pts.})}$$

$$\begin{aligned}
\mathscr{L}\{f(t)\} &= \mathscr{L}\{t\} - \mathscr{L}\{t\,u_{2\pi}(t)\} + \mathscr{L}\{\sin t\, u_{2\pi}(t)\} - \mathscr{L}\{\sin t\, u_{4\pi}(t)\} \\
&= \frac{1}{s^2} - \mathscr{L}\{((t-2\pi)+2\pi)\,u_{2\pi}(t)\} + \mathscr{L}\{\sin(t-2\pi)\,u_{2\pi}(t)\} \\
&\qquad - \mathscr{L}\{\sin(t-4\pi)\,u_{4\pi}(t)\} \\
&= \frac{1}{s^2} - e^{-2\pi s}\mathscr{L}\{t+2\pi\} + e^{-2\pi s}\mathscr{L}\{\sin t\} - e^{-4\pi s}\mathscr{L}\{\sin t\} \\
&= \underbrace{\frac{1}{s^2}}_{(0.5\text{ pts.})} \underbrace{- e^{-2\pi s}\left(\frac{1}{s^2} + \frac{2\pi}{s}\right)}_{(1.5\text{ pts.})} + \underbrace{e^{-2\pi s}\frac{1}{s^2+1}}_{(1.5\text{pts.})} \underbrace{- e^{-4\pi s}\frac{1}{s^2+1}}_{(1.5\text{ pts.})}
\end{aligned}$$

3. [10 points] Solve the differential equation:

$$y^{(5)} - 16y' = 0.$$

Characteristic Equation: $m^5 - 16m = 0$ (2pts.)

$m(m^4 - 16) = 0$

$m(m^2 - 4)(m^2 + 4) = 0$

$\boxed{m_1 = 0;\ m_2 = 2;\ m_3 = -2;\ m_{4,5} = \pm 2i}$ (5pts.)

$\Rightarrow \boxed{y = C_1 + C_2 e^{2x} + C_3 e^{-2x} + C_4 \sin(2x) + C_5 \cos(2x)}$ (3pts.)

4. [10 points] Find the matrix exponential e^{tA} for the matrix:

$$A = \begin{pmatrix} -1 & 1 \\ 0 & 2 \end{pmatrix}.$$

Solve $\vec{x}' = A\vec{x}$:

2pts. Eigenvalues: $\begin{vmatrix} -1-\lambda & 1 \\ 0 & 2-\lambda \end{vmatrix} = (\lambda+1)(\lambda-2)$ $\quad \lambda_1 = -1 \quad \lambda_2 = 2$

2pts. Eigenvectors:

$\lambda_1 = -1$: $\left(\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 3 & 0 \end{array}\right) \Rightarrow v_2 = 0 \Rightarrow \vec{v_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\lambda_2 = 2$: $\left(\begin{array}{cc|c} -3 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow 3v_1 = v_2 \Rightarrow \vec{v_2} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

Solution: $c_1 e^{-t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{2t}\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ or Fundamental Set: $\left\{ e^{-t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}; e^{2t}\begin{pmatrix} 1 \\ 3 \end{pmatrix} \right\}$

2pts. $\Rightarrow$ Fundamental Matrix: $\Phi(t) = \begin{pmatrix} e^{-t} & e^{2t} \\ 0 & 3e^{2t} \end{pmatrix}$

2pts. $\Rightarrow \Phi^{-1}(t) = \dfrac{1}{3e^{t}}\begin{pmatrix} 3e^{2t} & -e^{2t} \\ 0 & e^{-t} \end{pmatrix} \Rightarrow \Phi^{-1}(0) = \frac{1}{3}\begin{pmatrix} 3 & -1 \\ 0 & 1 \end{pmatrix}$

1pt. $\Rightarrow e^{tA} = \Psi(t) = \Phi(t)\Phi^{-1}(0) = \frac{1}{3}\begin{pmatrix} e^{-t} & e^{2t} \\ 0 & 3e^{2t} \end{pmatrix}\begin{pmatrix} 3 & -1 \\ 0 & 1 \end{pmatrix}$

1pt. $\Rightarrow \boxed{e^{tA} = \frac{1}{3}\begin{pmatrix} 3e^{-t} & -e^{-t}+e^{2t} \\ 0 & 3e^{2t} \end{pmatrix}}$

5. [10 points] Solve the differential equation:

$$y' + y^2 \cos x = 0,$$

subject to the following initial conditions:

a). $y(0) = \frac{1}{2}$.

$$\frac{dy}{dx} = -y^2 \cos x$$

If $y \neq 0$: $-\frac{1}{y^2} dy = \cos x \, dx$

$$\frac{1}{y} = \sin x + C \Rightarrow \boxed{y = \frac{1}{\sin x + C}} \text{ or } \boxed{y = 0}$$

$y = 0$ solution? Yes

$$y(0) = \frac{1}{C} = \frac{1}{2} \Rightarrow C = 2$$

$$\Rightarrow \boxed{y = \frac{1}{\sin x + 2}}$$

b). $y(0) = 0$.

$y(0) = 0$

$y(x) = \frac{1}{\sin x + C} \Rightarrow y(0) = \frac{1}{C}$ can never be 0

$\Rightarrow \boxed{y = 0}$ only solution to this IVP

Grading Rubric:
- Separation: 1pt.
- Treating $y=0$ separately: 1pt.
- Integrals: 2pts.
- Solution $y = \frac{1}{\sin x + C}$: 1pt.
- Sol. to IVP (a): 2pts.
- Sol. to IVP (b): 3pts.

6. [10 points] Find a particular solution to the equation:

$$y'' - \frac{1}{x}y' + \frac{1}{x^2}y = \ln x,\ x > 0,$$

given that

$$y_1(x) = x \quad \text{and} \quad y_2(x) = x\ln x$$

are two solutions of the associated homogeneous equation $y'' - \frac{1}{x}y' + \frac{1}{x^2}y = 0$.

$$W = \begin{vmatrix} x & x\ln x \\ 1 & 1+\ln x \end{vmatrix} = x + x\ln x - x\ln x = \boxed{x} \quad \text{1pt.}$$

$$W_1 = \begin{vmatrix} 0 & x\ln x \\ \ln x & 1+\ln x \end{vmatrix} = \boxed{-x\ln^2 x} \text{ 1pt.} \Rightarrow \boxed{u_1' = -\ln^2 x} \quad \text{1pt.}$$

$$W_2 = \begin{vmatrix} x & 0 \\ 1 & \ln x \end{vmatrix} = \boxed{x\ln x} \text{ 1pt.} \Rightarrow \boxed{u_2' = \ln x} \quad \text{1pt.}$$

$$\Rightarrow u_1 = -\int \ln^2 x\,dx = -\int x'\ln^2 x\,dx = -x\ln^2 x + \int x \cdot 2\ln x \cdot \frac{1}{x}\,dx$$

$$= -x\ln^2 x + 2\int \ln x\,dx$$

$$= \boxed{-x\ln^2 x + 2x\ln x - 2x} \quad \text{2pts.}$$

$$\Rightarrow u_2 = \int \ln x\,dx \Rightarrow \boxed{u_2 = x\ln x - x} \quad \text{2pts.}$$

$$\Rightarrow y_p = \boxed{\left(-x\ln^2 x + 2x\ln x - 2x\right)x + (x\ln x - x)x\ln x} \quad \text{1pt.}$$

$$= -\cancel{x^2\ln^2 x} + 2x^2\ln x - 2x^2 + \cancel{x^2\ln^2 x} - x^2\ln x$$

$$\text{or } = \boxed{x^2\ln x - 2x^2}$$

7. [10 points] Solve the differential equation:

$$x^2y'' - xy' + 2y = 0, \; x > 0.$$

(1pt). Cauchy – Euler with $a=1$; $b=-1$; $c=2$

Char. Eqn.: $m^2 - 2m + 2 = 0$ (2pts.)

$\Delta = 4 - 8 = -4$

$m = \frac{2 \pm 2i}{2} = 1 \pm i$ (3pts.)

$$\boxed{y = x\left(c_1 \cos(\ln x) + c_2 \sin(\ln x)\right)}$$ (4pts.)

8. [10 points] Find a solution to the equation:

$$y(t) = e^{2t} - \underbrace{\int_0^t (t-\tau)y(\tau)\,d\tau}_{t * y(t)}.$$

2 pts. - recognize convolution

$$Y(s) = \frac{1}{s-2} - \frac{1}{s^2}Y(s)$$

1 pt. - Take Laplace of both sides

$$\frac{s^2+1}{s^2}Y(s) = \frac{1}{s-2}$$

$$Y(s) = \frac{s^2}{(s-2)(s^2+1)}$$

1 pt. - Obtain $Y(s)$

Partial Fraction Decomposition:

$$\frac{s^2}{(s-2)(s^2+1)} = \frac{A}{s-2} + \frac{Bs+C}{s^2+1}$$

1 pt. - Setup

4 pts.

$A = 4/5$

$$\frac{4}{5}s^2 + \frac{4}{5} + Bs^2 + Cs - 2Bs - 2C = s^2$$

$$\left(\frac{4}{5}+B\right)s^2 + (C-2B)s + \left(\frac{4}{5}-2C\right) = s^2$$

$B = 1/5$ $C = 2/5$

1 pt. x each coefficient

$$\Rightarrow y(t) = \frac{4}{5}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} + \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + \frac{2}{5}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$\Rightarrow \boxed{y(t) = \frac{4}{5}e^{2t} + \frac{1}{5}\cos t + \frac{2}{5}\sin t}$$

2 pts. - Solution.

[1 pt. bonus:] What is this type of equation called? Volterra Integral Equation

9. [10 points]

a). [3 points] Find: $\mathscr{L}^{-1}\left\{\dfrac{1}{s^2-4s+13}\right\} = \mathscr{L}^{-1}\left\{\dfrac{1}{(s-2)^2+9}\right\}$

$$= \mathscr{L}^{-1}\left\{\frac{1}{s^2+9}\Big|_{s\to s-2}\right\} = e^{2t}\,\frac{1}{3}\sin(3t)$$

Completing Square – (2pts.)
Answer : (1pt.)

b). [4 points] Find: $\mathscr{L}^{-1}\left\{\dfrac{3s-12}{s^2-4s+13}\right\} = \mathscr{L}^{-1}\left\{\dfrac{3s-12}{(s-2)^2+9}\right\}$

$$= \mathscr{L}^{-1}\left\{\frac{3(s-2)}{(s-2)^2+9}\right\} + \mathscr{L}^{-1}\left\{\frac{-6}{(s-2)^2+9}\right\}$$

$$= 3\mathscr{L}^{-1}\left\{\frac{s}{s^2+9}\Big|_{s\to s-2}\right\} - 6\mathscr{L}^{-1}\left\{\frac{1}{s^2+9}\Big|_{s\to s-2}\right\}$$

$$= 3e^{2t}\cos(3t) - 2e^{2t}\sin(3t)$$

Correct Separation: (3 pts.)
Answer : (1pt.)

c). [3 points] Solve the initial value problem:

$$y'' - 4y' + 13y = \delta(t-3); \quad y(0) = 3, y'(0) = 0.$$

Take Laplace of both sides :

$$s^2Y(s) - 3s - 4sY(s) + 12 + 13Y(s) = e^{-3s} \quad (1\text{pt.})$$

$$(s^2-4s+13)Y(s) = 3s-12+e^{-3s}$$

$$Y(s) = \frac{3s-12}{s^2-4s+13} + \frac{e^{-3s}}{s^2-4s+13} \quad (1\text{pt.})$$

(1pt.) applying Translation Theorem to (a).

$$\Rightarrow y(t) = 3e^{2t}\cos(3t) - 2e^{2t}\sin(3t) + \frac{1}{3}e^{2(t-3)}\sin(3t-9)u_3(t)$$

10. [10 points] Given that

$$\mathbf{x_1} = e^{2t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad \text{and} \quad \mathbf{x_2} = e^{7t}\begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

are solutions to a homogeneous linear system $\mathbf{x}' = A\mathbf{x}$ for a real 2×2 matrix A, solve the initial value problem:

$$\mathbf{x}' = A\mathbf{x}; \ \mathbf{x}(0) = \begin{pmatrix} 3 \\ 4 \end{pmatrix}.$$

$$\vec{X} = C_1 e^{2t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} + C_2 e^{7t}\begin{pmatrix} 1 \\ 3 \end{pmatrix}$$ 2pts.

$$\vec{X}(0) = C_1\begin{pmatrix} 1 \\ -2 \end{pmatrix} + C_2\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} C_1 + C_2 \\ -2C_1 + 3C_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$ 2pts.

$$\Rightarrow 5C_2 = 10 \Rightarrow C_2 = 2 \Rightarrow C_1 = 1$$ 2pts. 2pts.

$$\Rightarrow \boxed{\vec{X}(t) = e^{2t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} + 2e^{7t}\begin{pmatrix} 1 \\ 3 \end{pmatrix}}$$ 2pts.

11. [10 points] Suppose that

$$\Phi(t) = \begin{pmatrix} e^t & 2e^{2t} \\ 0 & e^{2t} \end{pmatrix}$$

is a fundamental matrix for the linear system $\mathbf{x}' = A\mathbf{x}$, where A is a real 2×2 matrix. Find a particular solution to the linear system:

$$\mathbf{x}' = A\mathbf{x} + \begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix}.$$

1pt. $\vec{X}_p = \Phi(t) \int \Phi^{-1}(t)\vec{g}(t)\,dt$

3pts. $\Phi^{-1}(t) = \frac{1}{e^{3t}} \begin{pmatrix} e^{2t} & -2e^{2t} \\ 0 & e^t \end{pmatrix} = \begin{pmatrix} e^{-t} & -2e^{-t} \\ 0 & e^{-2t} \end{pmatrix} \cdot \begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix}$

2pts. $\Rightarrow \Phi^{-1}(t)\vec{g}(t) = \begin{pmatrix} e^t - 2e^t \\ 1 \end{pmatrix} = \begin{pmatrix} -e^t \\ 1 \end{pmatrix}$

2pts. $\Rightarrow \int \Phi^{-1}(t)\vec{g}(t)\,dt = \begin{pmatrix} -e^t \\ t \end{pmatrix}$

$\Rightarrow \vec{X}_p = \begin{pmatrix} e^t & 2e^{2t} \\ 0 & e^{2t} \end{pmatrix} \begin{pmatrix} -e^t \\ t \end{pmatrix} = \begin{pmatrix} -e^{2t} + 2te^{2t} \\ te^{2t} \end{pmatrix}$

2pts. $\Rightarrow \boxed{\vec{X}_p = \begin{pmatrix} 2te^{2t} - e^{2t} \\ te^{2t} \end{pmatrix}}$

12. [10 points] Find the general solution to the differential equation:

$$\cos(xy) - xy\sin(xy) - x^2\sin(xy)y' = 0.$$

(An implicit solution is fine.)

$$\left(\cos(xy) - xy\sin(xy)\right)dx - x^2\sin(xy)\,dy = 0$$

$$\left.\begin{aligned} \frac{\partial M}{\partial y} &= -x\sin(xy) - x\sin(xy) - x^2y\cos(xy) \\ \frac{\partial N}{\partial x} &= -2x\sin(xy) - x^2y\cos(xy) \end{aligned}\right\} \Rightarrow \underline{\text{Exact}} \quad \text{2pts.}$$

Potential: 7pts.

$$\frac{\partial f}{\partial y} = -x^2\sin(xy) \Rightarrow f(x,y) = x\cos(xy) + g(x)$$

$$\left.\begin{aligned} \Rightarrow \frac{\partial f}{\partial x} &= \cos(xy) - xy\sin(xy) + g'(x) \\ &= \cos(xy) - xy\sin(xy) \end{aligned}\right\} \Rightarrow \begin{aligned} g'(x) &= 0 \\ g(x) &= 0 \end{aligned}$$

$$\boxed{x\cos(xy) = C} \quad \text{1pt.}$$

13. [10 points] Consider the autonomous equation:

$$y' = (y^2 + 4)(y - 1)(y + 2).$$

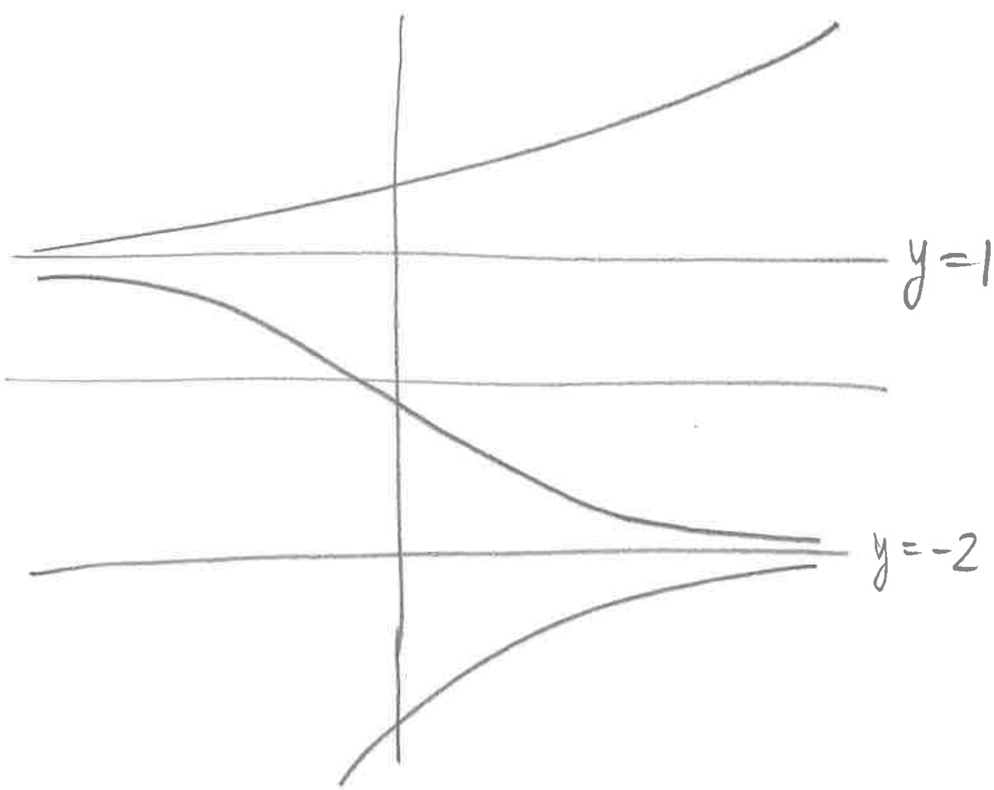
a). [3 points] What are the equilibrium solutions of this equation?

$y = 1$; $y = -2$

b). [4 points] Draw the phase portrait of this equation.

y		-2		1	
y^2+4	+	+	+	+	+
$y-1$	−	−	−	0	+
$y+2$	−	0	+	+	+
y'	+	0	−	0	+

c). [3 points] Draw a picture of what the solution curves for this equation might look like.



14. [10 points] Find the inverse Laplace transforms of the following functions:

a). [3 points] $\mathscr{L}^{-1}\left\{\frac{s+1}{s^2+4}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}$

$= \cos(2t) + \frac{1}{2}\sin(2t)$

b). [4 points] $\mathscr{L}^{-1}\left\{\frac{2s-4}{(s-1)(s+2)}\right\} = -\frac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \frac{8}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}$

$= \boxed{-\frac{2}{3}e^{t} + \frac{8}{3}e^{-2t}}$

$\frac{2s-4}{(s-1)(s+2)} = \frac{A}{s-1} + \frac{B}{s+2}$

$A = -\frac{2}{3}$ $B = \frac{+8}{+3}$

c). [3 points] $\mathscr{L}^{-1}\left\{\frac{1}{s^2+9}e^{-s}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\}\Big|_{t\to t-1} u_1(t)$

$= \frac{1}{3}\sin(3t-3)u_1(t).$

Bonus Problem: [10 points] Solve the differential equation for the function $y(x)$:

$$x^2y' + 2xy = x^4y^2 + 1.$$

Give an explicit answer.

$u = x^2 y \Rightarrow u' = 2xy + x^2 y'$

$\Rightarrow u' = u^2 + 1$ (autonomous)

$\frac{1}{u^2+1} du = dx$

$\arctan u = x + C$

$u = \tan(x+C)$

$\boxed{x^2 y = \tan(x+C)}$