Name: Solutions

___ November 4^{th}, 2015.
Math 2552; Sections L1 – L4.
Georgia Institute of Technology
Exam 2

I commit to uphold the ideals of honor and integrity by refusing to betray the trust bestowed upon me as a member of the Georgia Tech community. By signing my name below I pledge that I have neither given nor received help on this exam.

Pledged: ______________________

Problem	Possible Score	Earned Score
1	15	
2	18	
3	30	
4	25	
5	12	
Total	100	

Remember that you must SHOW YOUR WORK to receive credit!

Good luck!

Laplace transforms of some basic functions

$\mathscr{L}\{1\} = \frac{1}{s};\ s > 0$

$\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}};\ s > 0$

$\mathscr{L}\{e^{kt}\} = \frac{1}{s-k};\ s > k$

$\mathscr{L}\{\sin(kt)\} = \frac{k}{s^2+k^2};\ s > 0$

$\mathscr{L}\{\cos(kt)\} = \frac{s}{s^2+k^2};\ s > 0$

$\mathscr{L}\{\sinh(kt)\} = \frac{k}{s^2-k^2};\ s > |k|$

$\mathscr{L}\{\cosh(kt)\} = \frac{s}{s^2-k^2};\ s > |k|$

$\mathscr{L}\{u_a(t)\} = \frac{e^{-as}}{s};\ s > 0$

$\mathscr{L}\{\delta(t-t_0)\} = e^{-st_0}$

$\mathscr{L}\{\delta(t)\} = 1$

Inverse Laplace transforms of some basic functions

$\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} = 1$

$\mathscr{L}^{-1}\left\{\frac{1}{s^n}\right\} = \frac{1}{(n-1)!}t^{n-1}$

$\mathscr{L}^{-1}\left\{\frac{1}{s-k}\right\} = e^{kt}$

$\mathscr{L}^{-1}\left\{\frac{1}{s^2+k^2}\right\} = \frac{1}{k}\sin(kt)$

$\mathscr{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\} = \cos(kt)$

$\mathscr{L}^{-1}\left\{\frac{1}{s^2-k^2}\right\} = \frac{1}{k}\sinh(kt)$

$\mathscr{L}^{-1}\left\{\frac{s}{s^2-k^2}\right\} = \cosh(kt)$

$\mathscr{L}^{-1}\left\{\frac{e^{-as}}{s}\right\} = u_a(t)$

$\mathscr{L}^{-1}\left\{e^{-st_0}\right\} = \delta(t-t_0)$

$\mathscr{L}^{-1}\{1\} = \delta(t)$

Properties of the Laplace and Inverse Laplace transform

Translation Theorem I:

$$\mathscr{L}\left\{e^{kt}f(t)\right\} = F(s-k) = \mathscr{L}\{f(t)\}|_{s\to s-k}$$

$$\mathscr{L}^{-1}\{F(s-k)\} = \mathscr{L}^{-1}\left\{F(s)|_{s\to s-k}\right\} = e^{kt}\mathscr{L}^{-1}\{F(s)\} = e^{kt}f(t)$$

Translation Theorem II:

$$\mathscr{L}\left\{f(t-a)u_a(t)\right\} = e^{-as}F(s) = e^{-as}\mathscr{L}\{f(t)\}$$

$$\mathscr{L}^{-1}\{e^{-as}F(s)\} = f(t-a)u_a(t) = \mathscr{L}^{-1}\{F(s)\}\big|_{t\to t-a}\, u_a(t)$$

Derivatives of Laplace Transforms:

$$\mathscr{L}\left\{t^n f(t)\right\} = (-1)^n\frac{d^n}{ds^n}F(s)$$

$$\mathscr{L}^{-1}\left\{F^{(n)}(s)\right\} = (-1)^n t^n f(t)$$

Laplace Transform of Periodic Functions:
If $f(t)$ is piecewise continuous on $[0,\infty)$, of exponential order, and periodic with period T:

$$\mathscr{L}\{f(t)\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)\,dt$$

Laplace Transforms of Derivatives:

$$\begin{aligned}
\mathscr{L}\{y'\} &= sY(s) - y(0)\\
\mathscr{L}\{y''\} &= s^2Y(s) - sy(0) - y'(0)\\
&\ \ \vdots\\
\mathscr{L}\{y^{(n)}(t)\} &= s^nY(s) - s^{n-1}y(0) - s^{n-2}y'(0) - \ldots - sy^{(n-2)}(0) - y^{(n-1)}(0)
\end{aligned}$$

Convolution Theorem:

$$\mathscr{L}\{f(t)\star g(t)\} = F(s)G(s) = \mathscr{L}\{f(t)\}\mathscr{L}\{g(t)\}$$

$$\mathscr{L}^{-1}\{F(s)G(s)\} = f(t)\star g(t)$$

Dirac Delta Function:

$$\int_0^\infty f(t)\delta(t-t_0)\,dt = f(t_0)$$

$$(f\star\delta)(t) = f(t)$$

$(3 \text{ pts. each}) \times 5 = 15 \text{ pts.}$

1. Consider the linear equation:

$$y^{(4)} - 7y'' - 18y = g(x).$$

The complementary solution is:

$$y_c = c_1 e^{3x} + c_2 e^{-3x} + c_3 \sin(\sqrt{2}x) + c_4 \cos(\sqrt{2}x).$$

For each of the functions $g(x)$ below, write down the correct guess for a particular solution y_p, if one were to use the method of Undetermined Coefficients to solve the equation:

a). $g(x) = x\sin(\sqrt{2}x)$.

$$y_p = \left((Ax+B)\sin(\sqrt{2}x) + (Cx+D)\cos(\sqrt{2}x)\right)\cdot X$$
$$= (AX^2+BX)\sin(\sqrt{2}X) + (CX^2+DX)\cos(\sqrt{2}X)$$

b). $g(x) = x^2 e^x$.

$$y_p = (AX^2+BX+C)e^X$$

c). $g(x) = x^2 e^{3x}$.

$$y_p = (AX^2+BX+C)e^{3X}\cdot X = (AX^3+BX^2+CX)e^{3X}$$

d). $g(x) = e^{3x}\sin(\sqrt{2}x)$.

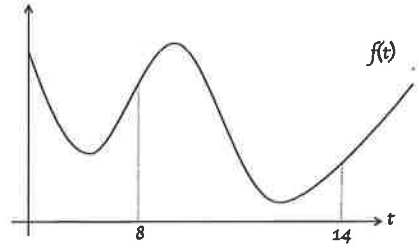
$$y_p = Ae^{3X}\sin(\sqrt{2}X) + Be^{3X}\cos(\sqrt{2}X)$$

e). $g(x) = x. + Xe^{-3X}$

$$y_p = (AX+B) + (CX+D)e^{-3X}\cdot X$$
$$= (AX+B) + (CX^2+DX)e^{-3X}$$

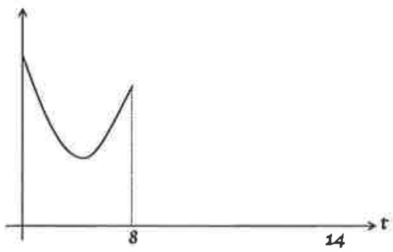
(3 pts. each) × 6 = 18 pts.

2. Consider the function $f(t)$, for $t \in [0, \infty)$, graphed to the right. Match each of the following graphs (obtained by various translations and/or "turning off" of the graph of f) with the expressions below.

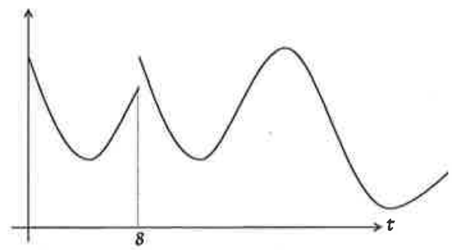


Place the number of the correct expression in the box under each graph.

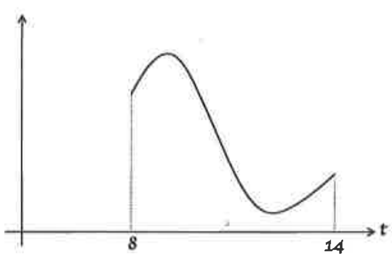
1. $f(t)\big(1-u_8(t)\big)$
2. $f(t)\big(1-u_8(t)\big)+f(t-8)u_8(t)$
3. $f(t-8)\big(u_8(t)-u_{14}(t)\big)$
4. $f(t-8)u_8(t)$
5. $f(t)\big(u_8(t)-u_{14}(t)\big)$
6. $f(t)u_8(t)$



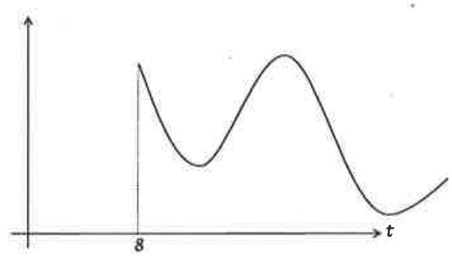
$\boxed{1}$



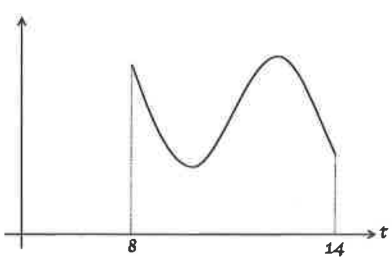
$\boxed{2}$



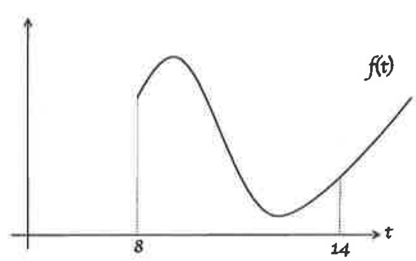
$\boxed{5}$



$\boxed{4}$



$\boxed{3}$



$\boxed{6}$

(6 pts. each) × 5 = 30 pts.

3. Compute the following:

a). $\mathscr{L}^{-1}\left\{\frac{s}{s^2+4s+13}\right\} = \mathscr{L}^{-1}\left\{\frac{s}{(s+2)^2+9}\right\}$

(2pts) - completing square

$$= \mathscr{L}^{-1}\left\{\frac{(s+2)-2}{(s+2)^2+9}\right\}$$

$$= \mathscr{L}^{-1}\left\{\frac{s}{s^2+9}\Big|_{s\to s+2}\right\} - 2\mathscr{L}^{-1}\left\{\frac{1}{s^2+9}\Big|_{s\to s+2}\right\}$$

$$= \boxed{e^{-2t}\cos(3t) - 2e^{-2t}\frac{1}{3}\sin(3t)}$$

(2pts.) (2pts.)

b). $\mathscr{L}^{-1}\left\{\frac{s-3}{s^2+7}\right\} = \mathscr{L}^{-1}\left\{\frac{s}{s^2+7}\right\} - 3\mathscr{L}^{-1}\left\{\frac{1}{s^2+7}\right\}$

$$= \boxed{\cos(\sqrt{7}t) - \frac{3}{\sqrt{7}}\sin(\sqrt{7}t)}$$

(3pts.) (3pts.)

c). $\mathscr{L}^{-1}\left\{\frac{2s+3}{s(s+3)}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{s} + \frac{1}{s+3}\right\} = \boxed{1+e^{-3t}}$

$$\frac{2s+3}{s(s+3)} = \frac{A}{s} + \frac{B}{s+3}$$

$$\frac{2s+3}{s+3}\Big|_{s=0} = A \Rightarrow A=1$$

$$\frac{2s+3}{s}\Big|_{s=-3} = B \Rightarrow B=1$$

or

$$\frac{2s+3}{s(s+3)} = \frac{s+(s+3)}{s(s+3)} = \frac{1}{s+3} + \frac{1}{s}$$

(4pts.) - partial fraction decomp.

(2pts.) - final answer

d). $\mathscr{L}\{f(t)\}$, where $f(t) = \begin{cases} 0, & 0 \le t < 1 \\ t^2, & 1 \le t < 2 \\ 0, & t \ge 2. \end{cases}$ $= t^2(u_1(t) - u_2(t))$

(2 pts.) - writing f(t) in terms of unit step functions

$$\mathscr{L}\{f(t)\} = \mathscr{L}\{t^2 u_1(t) - t^2 u_2(t)\}$$
$$= \mathscr{L}\{((t-1)+1)^2 u_1(t)\} - \mathscr{L}\{((t-2)+2)^2 u_2(t)\}$$
$$= e^{-s}\mathscr{L}\{(t+1)^2\} - e^{-2s}\mathscr{L}\{(t+2)^2\}$$
$$= e^{-s}\mathscr{L}\{t^2+2t+1\} - e^{-2s}\mathscr{L}\{t^2+4t+4\}$$
$$= e^{-s}\left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right) - e^{-2s}\left(\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}\right)$$

(4 pts.) - $\mathscr{L}\{f(t)\}$

e). $\mathscr{L}\{y(t)\}$, where $y(t)$ is the solution to the initial value problem:

$$y'' + 4y' + 6y = 1 + e^{-t}; \;\; y(0) = y'(0) = 0.$$

$$\mathscr{L}\{y''+4y'+6y\} = \mathscr{L}\{1+e^{-t}\}$$
$$s^2 Y(s) + 4s\,Y(s) + 6\,Y(s) = \frac{1}{s} + \frac{1}{s+1}$$
$$Y(s) = \boxed{\left(\frac{1}{s} + \frac{1}{s+1}\right)\frac{1}{s^2+4s+6}}$$
$$= \boxed{\frac{2s+1}{s(s+1)(s^2+4s+6)}}$$

(1pt.) - taking Laplace of both sides

(3pts.) - $\mathscr{L}\{e^{-t}+1\}$

(2pts.) - final answer

4. Find the general solution to the differential equation:

$$y'' + 4y = \frac{1}{\cos(2x)}.$$

(8 pts.) Complementary Solution:

$m^2+4=0$ (4 pts.)

$m=\pm 2i$

$$\boxed{y_c = C_1\cos(2x)+C_2\sin(2x)}$$ (4 pts.)

(16 pts.) Particular Solution:

$$W=\begin{vmatrix} \cos(2x) & \sin(2x) \\ -2\sin(2x) & 2\cos(2x) \end{vmatrix} = 2\cos^2(2x)+2\sin^2(2x)=2$$ (3 pts.)

$$W_1=\begin{vmatrix} 0 & \sin(2x) \\ \frac{1}{\cos(2x)} & 2\cos(2x) \end{vmatrix} = -\frac{\sin(2x)}{\cos(2x)}$$ (3 pts.)

$$u_1' = -\frac{\sin(2x)}{2\cos(2x)} \Rightarrow \boxed{u_1 = \frac{1}{4}\ln|\cos(2x)|}$$ (3 pts.)

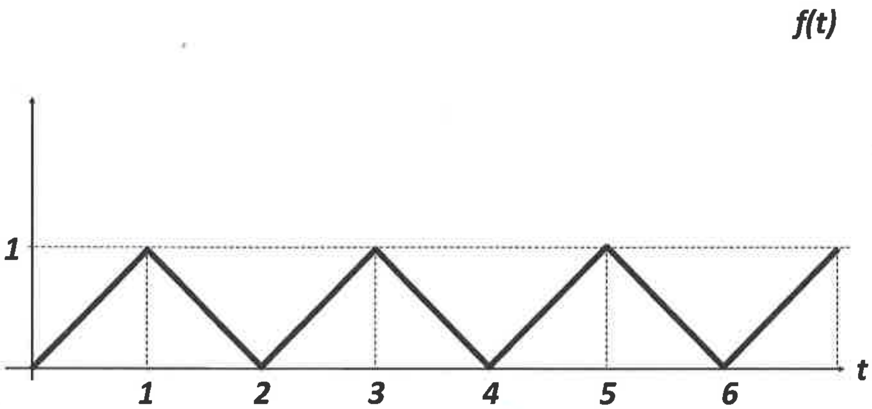
$$W_2=\begin{vmatrix} \cos(2x) & 0 \\ -2\sin(2x) & \frac{1}{\cos(2x)} \end{vmatrix} = 1$$ (3 pts.)

$$u_2' = \frac{1}{2} \Rightarrow \boxed{u_2 = \frac{1}{2}x}$$ (3 pts.)

$$\boxed{y_p = \frac{1}{4}\cos(2x)\ln|\cos(2x)| + \frac{1}{2}x\sin(2x)}$$ (1 pt.)

(1 pt.) General Solution: $\boxed{y = y_c + y_p}$

5. Find the Laplace transform of the periodic function $f(t)$, pictured below:



Period $\boxed{T=2}$ (2pts.)

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-2s}} \mathcal{L}\overbrace{\left\{ t(1-u_1(t)) + (2-t)(u_1(t)-u_2(t)) \right\}}^{\text{(5pts.)}}$$

$$= \frac{1}{1-e^{-2s}} \mathcal{L}\left\{ t + u_1(t)(2-2t) - (2-t)u_2(t) \right\}$$

$$= \frac{1}{1-e^{-2s}} \left(\frac{1}{s^2} - 2e^{-s}\mathcal{L}\{t\} + e^{-2s}\mathcal{L}\{t\} \right)$$

$$= \boxed{\frac{1}{1-e^{-2s}} \left(\frac{1}{s^2} - \frac{2e^{-s}}{s^2} + \frac{e^{-2s}}{s^2} \right)}$$

$$= \boxed{\frac{1-2e^{-s}+e^{-2s}}{s^2(1-e^{-2s})}}$$

(1pt.) (2pts.) (2pts.)