Name: ______________________________

November 4^{th}, 2015.
Math 2552; Sections F1 – F4.
Georgia Institute of Technology
Exam 2

I commit to uphold the ideals of honor and integrity by refusing to betray the trust bestowed upon me as a member of the Georgia Tech community. By signing my name below I pledge that I have neither given nor received help on this exam.

Pledged: ______________________________

Problem	Possible Score	Earned Score
1	15	
2	18	
3	30	
4	25	
5	12	
Total	100	

Remember that you must SHOW YOUR WORK to receive credit!

Good luck!

Laplace transforms of some basic functions

$\mathscr{L}\{1\} = \frac{1}{s};\ s > 0$

$\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}};\ s > 0$

$\mathscr{L}\{e^{kt}\} = \frac{1}{s-k};\ s > k$

$\mathscr{L}\{\sin(kt)\} = \frac{k}{s^2+k^2};\ s > 0$

$\mathscr{L}\{\cos(kt)\} = \frac{s}{s^2+k^2};\ s > 0$

$\mathscr{L}\{\sinh(kt)\} = \frac{k}{s^2-k^2};\ s > |k|$

$\mathscr{L}\{\cosh(kt)\} = \frac{s}{s^2-k^2};\ s > |k|$

$\mathscr{L}\{u_a(t)\} = \frac{e^{-as}}{s};\ s > 0$

$\mathscr{L}\{\delta(t-t_0)\} = e^{-st_0}$

$\mathscr{L}\{\delta(t)\} = 1$

Inverse Laplace transforms of some basic functions

$\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} = 1$

$\mathscr{L}^{-1}\left\{\frac{1}{s^n}\right\} = \frac{1}{(n-1)!}t^{n-1}$

$\mathscr{L}^{-1}\left\{\frac{1}{s-k}\right\} = e^{kt}$

$\mathscr{L}^{-1}\left\{\frac{1}{s^2+k^2}\right\} = \frac{1}{k}\sin(kt)$

$\mathscr{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\} = \cos(kt)$

$\mathscr{L}^{-1}\left\{\frac{1}{s^2-k^2}\right\} = \frac{1}{k}\sinh(kt)$

$\mathscr{L}^{-1}\left\{\frac{s}{s^2-k^2}\right\} = \cosh(kt)$

$\mathscr{L}^{-1}\left\{\frac{e^{-as}}{s}\right\} = u_a(t)$

$\mathscr{L}^{-1}\left\{e^{-st_0}\right\} = \delta(t-t_0)$

$\mathscr{L}^{-1}\left\{1\right\} = \delta(t)$

Properties of the Laplace and Inverse Laplace transform

Translation Theorem I:

$$\mathscr{L}\left\{e^{kt}f(t)\right\} = F(s-k) = \mathscr{L}\{f(t)\}|_{s\to s-k}$$

$$\mathscr{L}^{-1}\{F(s-k)\} = \mathscr{L}^{-1}\left\{F(s)|_{s\to s-k}\right\} = e^{kt}\mathscr{L}^{-1}\{F(s)\} = e^{kt}f(t)$$

Translation Theorem II:

$$\mathscr{L}\left\{f(t-a)u_a(t)\right\} = e^{-as}F(s) = e^{-as}\mathscr{L}\{f(t)\}$$

$$\mathscr{L}^{-1}\{e^{-as}F(s)\} = f(t-a)u_a(t) = \mathscr{L}^{-1}\{F(s)\}\Big|_{t\to t-a} u_a(t)$$

Derivatives of Laplace Transforms:

$$\mathscr{L}\left\{t^n f(t)\right\} = (-1)^n \frac{d^n}{ds^n}F(s)$$

$$\mathscr{L}^{-1}\left\{F^{(n)}(s)\right\} = (-1)^n t^n f(t)$$

Laplace Transform of Periodic Functions:
If $f(t)$ is piecewise continuous on $[0,\infty)$, of exponential order, and periodic with period T:

$$\mathscr{L}\{f(t)\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)\,dt$$

Laplace Transforms of Derivatives:

$$\begin{aligned}
\mathscr{L}\{y'\} &= sY(s) - y(0)\\
\mathscr{L}\{y''\} &= s^2Y(s) - sy(0) - y'(0)\\
&\vdots\\
\mathscr{L}\{y^{(n)}(t)\} &= s^nY(s) - s^{n-1}y(0) - s^{n-2}y'(0) - \ldots - sy^{(n-2)}(0) - y^{(n-1)}(0)
\end{aligned}$$

Convolution Theorem:

$$\mathscr{L}\{f(t)\star g(t)\} = F(s)G(s) = \mathscr{L}\{f(t)\}\mathscr{L}\{g(t)\}$$

$$\mathscr{L}^{-1}\{F(s)G(s)\} = f(t)\star g(t)$$

Dirac Delta Function:

$$\int_0^\infty f(t)\delta(t-t_0)\,dt = f(t_0)$$

$$(f\star\delta)(t) = f(t)$$

1. Consider the linear equation:

$$y''' + y'' - 2y = g(x).$$

The complementary solution is:

$$y_c = c_1 e^x + c_2 e^{-x} \sin x + c_3 e^{-x} \cos x.$$

For each of the functions $g(x)$ below, write down the correct guess for a particular solution y_p, if one were to use the method of Undetermined Coefficients to solve the equation:

a). $g(x) = x^2 e^x$.

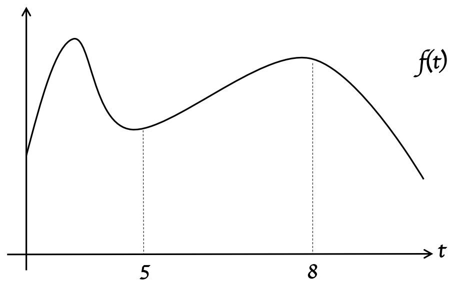
b). $g(x) = e^x \sin x$.

c). $g(x) = e^{-x} \sin x$.

d). $g(x) = x^2 + xe^x$.

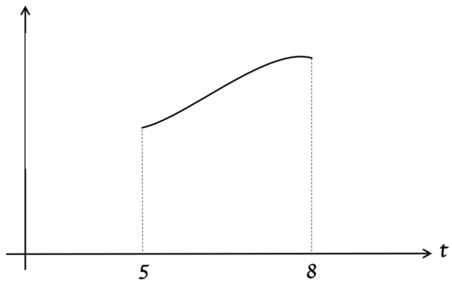
e). $g(x) = x \sin x$.

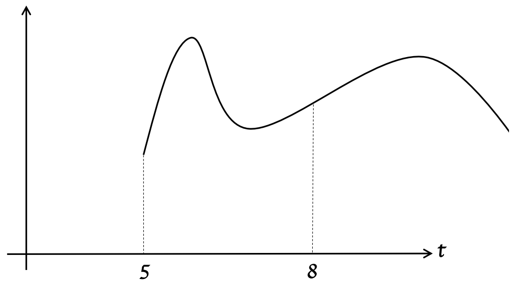
2. Consider the function $f(t)$, for $t \in [0, \infty)$, graphed to the right. Match each of the following graphs (obtained by various translations and/or "turning off" of the graph of f) with the expressions below.

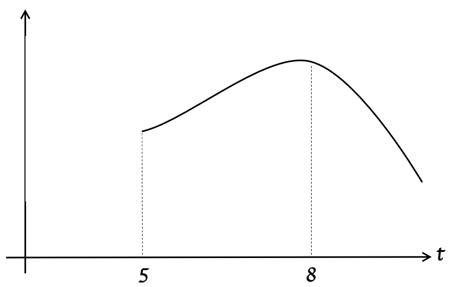


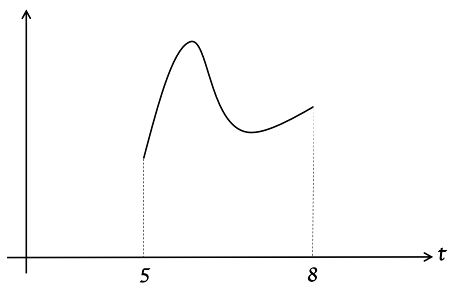
Place the number of the correct expression in the box under each graph.

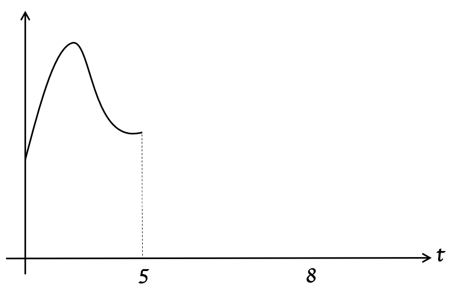
1. $f(t-5)\big(u_5(t) - u_8(t)\big)$
2. $f(t)\big(u_5(t) - u_8(t)\big)$
3. $f(t)\big(1 - u_5(t)\big) + f(t-5)\big(u_5(t) - u_8(t)\big)$
4. $f(t-5)u_5(t)$
5. $f(t)\big(1 - u_5(t)\big)$
6. $f(t)u_5(t)$

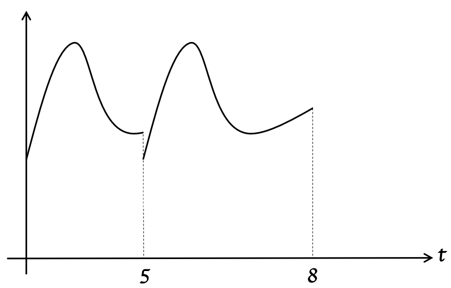












3. Compute the following:

a). $\mathscr{L}^{-1}\left\{\dfrac{s+2}{s^2+5}\right\}$

b). $\mathscr{L}^{-1}\left\{\dfrac{s}{s^2+4s+5}\right\}$

c). $\mathscr{L}^{-1}\left\{\dfrac{s+10}{(s+2)(s-6)}\right\}$

d). $\mathscr{L}\{f(t)\}$, where $f(t) = \begin{cases} t^2, & 0 \leq t < 2 \\ 6, & t \geq 2. \end{cases}$

e). $\mathscr{L}\{y(t)\}$, where $y(t)$ is the solution to the initial value problem:

$$y'' - y' = e^t \cos t; \;\; y(0) = y'(0) = 0.$$

4. Find the general solution to the differential equation:

$$y'' - 2y' + y = \frac{e^x}{1+x^2}.$$

5. Find the Laplace transform of the periodic function $f(t)$, pictured below: