Name: Solutions

November 4th, 2015.
Math 2552; Sections F1 – F4.
Georgia Institute of Technology
Exam 2

I commit to uphold the ideals of honor and integrity by refusing to betray the trust bestowed upon me as a member of the Georgia Tech community. By signing my name below I pledge that I have neither given nor received help on this exam.

Pledged: ______________________

Problem	Possible Score	Earned Score
1	15	
2	18	
3	30	
4	25	
5	12	
Total	100	

Remember that you must SHOW YOUR WORK to receive credit!

Good luck!

Laplace transforms of some basic functions

$\mathscr{L}\{1\} = \frac{1}{s};\ s > 0$

$\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}};\ s > 0$

$\mathscr{L}\{e^{kt}\} = \frac{1}{s-k};\ s > k$

$\mathscr{L}\{\sin(kt)\} = \frac{k}{s^2+k^2};\ s > 0$

$\mathscr{L}\{\cos(kt)\} = \frac{s}{s^2+k^2};\ s > 0$

$\mathscr{L}\{\sinh(kt)\} = \frac{k}{s^2-k^2};\ s > |k|$

$\mathscr{L}\{\cosh(kt)\} = \frac{s}{s^2-k^2};\ s > |k|$

$\mathscr{L}\{u_a(t)\} = \frac{e^{-as}}{s};\ s > 0$

$\mathscr{L}\{\delta(t-t_0)\} = e^{-st_0}$

$\mathscr{L}\{\delta(t)\} = 1$

Inverse Laplace transforms of some basic functions

$\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} = 1$

$\mathscr{L}^{-1}\left\{\frac{1}{s^n}\right\} = \frac{1}{(n-1)!}t^{n-1}$

$\mathscr{L}^{-1}\left\{\frac{1}{s-k}\right\} = e^{kt}$

$\mathscr{L}^{-1}\left\{\frac{1}{s^2+k^2}\right\} = \frac{1}{k}\sin(kt)$

$\mathscr{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\} = \cos(kt)$

$\mathscr{L}^{-1}\left\{\frac{1}{s^2-k^2}\right\} = \frac{1}{k}\sinh(kt)$

$\mathscr{L}^{-1}\left\{\frac{s}{s^2-k^2}\right\} = \cosh(kt)$

$\mathscr{L}^{-1}\left\{\frac{e^{-as}}{s}\right\} = u_a(t)$

$\mathscr{L}^{-1}\left\{e^{-st_0}\right\} = \delta(t-t_0)$

$\mathscr{L}^{-1}\{1\} = \delta(t)$

Properties of the Laplace and Inverse Laplace transform

Translation Theorem I:

$$\mathscr{L}\left\{e^{kt}f(t)\right\} = F(s-k) = \mathscr{L}\{f(t)\}|_{s\to s-k}$$

$$\mathscr{L}^{-1}\{F(s-k)\} = \mathscr{L}^{-1}\left\{F(s)|_{s\to s-k}\right\} = e^{kt}\mathscr{L}^{-1}\{F(s)\} = e^{kt}f(t)$$

Translation Theorem II:

$$\mathscr{L}\left\{f(t-a)u_a(t)\right\} = e^{-as}F(s) = e^{-as}\mathscr{L}\{f(t)\}$$

$$\mathscr{L}^{-1}\{e^{-as}F(s)\} = f(t-a)u_a(t) = \mathscr{L}^{-1}\{F(s)\}\big|_{t\to t-a}\,u_a(t)$$

Derivatives of Laplace Transforms:

$$\mathscr{L}\left\{t^n f(t)\right\} = (-1)^n\frac{d^n}{ds^n}F(s)$$

$$\mathscr{L}^{-1}\left\{F^{(n)}(s)\right\} = (-1)^n t^n f(t)$$

Laplace Transform of Periodic Functions:
If $f(t)$ is piecewise continuous on $[0,\infty)$, of exponential order, and periodic with period T:

$$\mathscr{L}\{f(t)\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)\,dt$$

Laplace Transforms of Derivatives:

$$\begin{aligned}
\mathscr{L}\{y'\} &= sY(s) - y(0)\\
\mathscr{L}\{y''\} &= s^2Y(s) - sy(0) - y'(0)\\
&\ \ \vdots\\
\mathscr{L}\{y^{(n)}(t)\} &= s^nY(s) - s^{n-1}y(0) - s^{n-2}y'(0) - \ldots - sy^{(n-2)}(0) - y^{(n-1)}(0)
\end{aligned}$$

Convolution Theorem:

$$\mathscr{L}\{f(t)\star g(t)\} = F(s)G(s) = \mathscr{L}\{f(t)\}\mathscr{L}\{g(t)\}$$

$$\mathscr{L}^{-1}\{F(s)G(s)\} = f(t)\star g(t)$$

Dirac Delta Function:

$$\int_0^\infty f(t)\delta(t-t_0)\,dt = f(t_0)$$

$$(f\star\delta)(t) = f(t)$$

(3 points each) x 5 = 15 pts.

1. Consider the linear equation:

$$y''' + y'' - 2y = g(x).$$

The complementary solution is:

$$y_c = c_1 e^x + c_2 e^{-x}\sin x + c_3 e^{-x}\cos x.$$

For each of the functions $g(x)$ below, write down the correct guess for a particular solution y_p, if one were to use the method of Undetermined Coefficients to solve the equation:

a). $g(x) = x^2 e^x$.

$$y_p = (Ax^2 + Bx + C)e^x \cdot \underline{\underline{x}} = (Ax^3 + Bx^2 + Cx)e^x$$

b). $g(x) = e^x \sin x$.

$$y_p = Ae^x \sin x + Be^x \cos x$$

c). $g(x) = e^{-x}\sin x$.

$$y_p = \left(Ae^{-x}\sin x + Be^{-x}\cos x\right) \cdot \underline{\underline{x}} = Axe^{-x}\sin x + Bxe^{-x}\cos x$$

d). $g(x) = x^2 + xe^x$

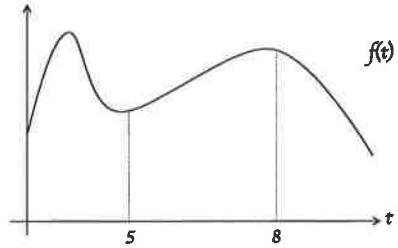
$$y_p = (Ax^2 + Bx + C) + (Dx + E)e^x \cdot \underline{\underline{x}} = (Ax^2 + Bx + C) + (Dx^2 + Ex)e^x$$

e). $g(x) = x \sin x$.

$$y_p = (Ax + B)\sin x + (Cx + D)\cos x$$

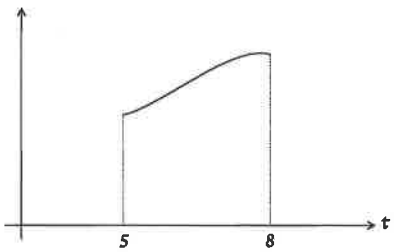
(3 pts. each) × 6 = 18 pts.

2. Consider the function $f(t)$, for $t \in [0, \infty)$, graphed to the right. Match each of the following graphs (obtained by various translations and/or "turning off" of the graph of f) with the expressions below.

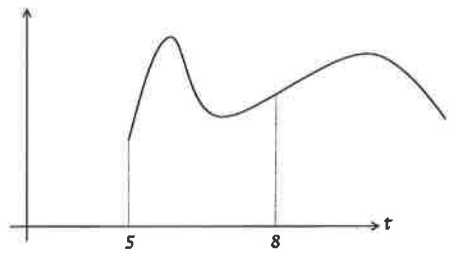


Place the number of the correct expression in the box under each graph.

1. $f(t-5)\big(u_5(t) - u_8(t)\big)$
2. $f(t)\big(u_5(t) - u_8(t)\big)$
3. $f(t)\big(1 - u_5(t)\big) + f(t-5)\big(u_5(t) - u_8(t)\big)$
4. $f(t-5)u_5(t)$
5. $f(t)\big(1 - u_5(t)\big)$
6. $f(t)u_5(t)$



$\boxed{2}$



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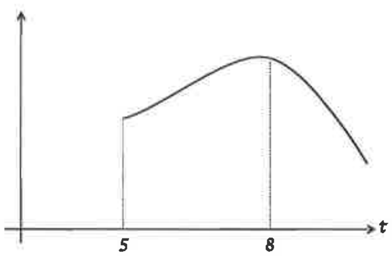

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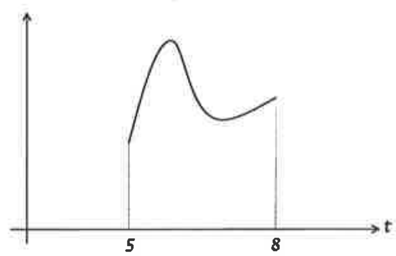
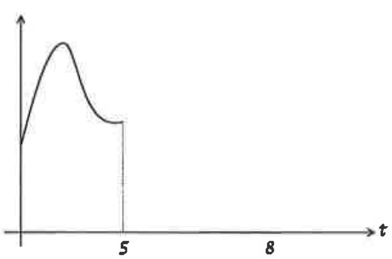
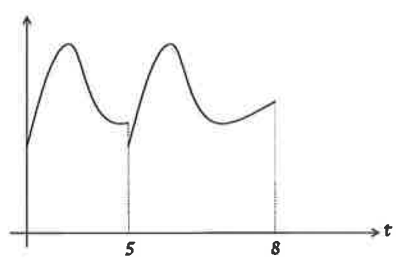


$\boxed{1}$



$\boxed{5}$



$\boxed{3}$

(6 pts. each) x5 = 30 pts.

3. Compute the following:

a). $\mathscr{L}^{-1}\left\{\frac{s+2}{s^2+5}\right\} = \mathscr{L}^{-1}\left\{\frac{s}{s^2+5}\right\} + 2\mathscr{L}^{-1}\left\{\frac{1}{s^2+5}\right\}$

$$= \boxed{\cos(\sqrt{5}t) + \frac{2}{\sqrt{5}}\sin(\sqrt{5}t).}$$

(3 pts.) (3pts.)

b). $\mathscr{L}^{-1}\left\{\frac{s}{s^2+4s+5}\right\} = \mathscr{L}^{-1}\left\{\frac{s}{(s+2)^2+1}\right\} = \mathscr{L}^{-1}\left\{\frac{(s+2)-2}{(s+2)^2+1}\right\}$

(2pts.) - completing the square

$$= \mathscr{L}^{-1}\left\{\frac{s}{s^2+1}\Big|_{s\to s+2}\right\} - 2\mathscr{L}^{-1}\left\{\frac{1}{s^2+1}\Big|_{s\to s+2}\right\}$$

$$= \boxed{e^{-2t}\cos(t) - 2e^{-2t}\sin(t)}$$

(2 pts.) (2pts.)

c). $\mathscr{L}^{-1}\left\{\frac{s+10}{(s+2)(s-6)}\right\} = \mathscr{L}^{-1}\left\{\frac{-1}{s+2} + \frac{2}{s-6}\right\} = \boxed{-e^{-2t} + 2e^{6t}}$

$$\frac{s+10}{(s+2)(s-6)} = \frac{A}{s+2} + \frac{B}{s-6}$$

$$\frac{s+10}{s-6}\Big|_{s=-2} = A \qquad A=-1$$

$$\frac{s+10}{s+2}\Big|_{s=6} = B \qquad B=2$$

or

$$2(s+2)-(s-6) = s+10$$

$$\Rightarrow \frac{s+10}{(s+2)(s-6)} = \frac{2(s+2)-(s-6)}{(s+2)(s-6)}$$

$$= \frac{2}{s-6} - \frac{1}{s+2}$$

(4pts) - partial fraction decomp.
(2pts.) - final answer

d). $\mathscr{L}\{f(t)\}$, where $f(t) = \begin{cases} t^2, & 0 \le t < 2 \\ 6, & t \ge 2. \end{cases}$

$$f(t) = t^2(1-u_2(t)) + 6u_2(t)$$
$$= t^2 - (t^2-6)u_2(t)$$

(3pts.) - writing $f(t)$ in terms of unit step functions

(3pts.) - $\mathscr{L}\{f(t)\}$

$$\mathscr{L}\{f(t)\} = \mathscr{L}\{t^2 - (t^2-6)u_2(t)\}$$
$$= \frac{2}{s^3} - \mathscr{L}\{[((t-2)+2)^2 - 6]u_2(t)\}$$
$$= \frac{2}{s^3} - e^{-2s}\mathscr{L}\{(t+2)^2 - 6\}$$
$$= \frac{2}{s^3} - e^{-2s}\mathscr{L}\{t^2+4t-2\}$$
$$= \boxed{\frac{2}{s^3} - e^{-2s}\left(\frac{2}{s^3} + \frac{4}{s^2} - \frac{2}{s}\right)}$$

e). $\mathscr{L}\{y(t)\}$, where $y(t)$ is the solution to the initial value problem:

$$y'' - y' = e^t \cos t; \quad y(0) = y'(0) = 0.$$

(1pt.) - taking Laplace of both sides

(3pts.) - $\mathscr{L}\{e^t \cos t\}$

(2pts.) - final answer

$$\mathscr{L}\{y'' - y'\} = \mathscr{L}\{e^t \cos t\}$$
$$s^2 Y(s) - sY(s) = \mathscr{L}\{\cos t\}\big|_{s \to s-1}$$
$$(s^2 - s)Y(s) = \frac{s-1}{(s-1)^2+1}$$
$$Y(s) = \frac{s-1}{((s-1)^2+1)s(s-1)} = \boxed{\frac{1}{s((s-1)^2+1)}}$$

4. Find the general solution to the differential equation:

$$y'' - 2y' + y = \frac{e^x}{1+x^2}.$$

(8pts) Complementary Solution:

$m^2 - 2m + 1 = 0$

$(m-1)^2 = 0$ (4pts.)

$m_1 = m_2 = 1$

$\boxed{y_c = c_1 e^x + c_2 x e^x}$ (4pts.)

(16pts.) Particular Solution:

$$W = \begin{vmatrix} e^x & xe^x \\ e^x & (x+1)e^x \end{vmatrix} = e^{2x}$$ (3pts.)

$$W_1 = \begin{vmatrix} 0 & xe^x \\ \frac{e^x}{1+x^2} & (x+1)e^x \end{vmatrix} = -\frac{xe^{2x}}{1+x^2}$$ (3pts.)

$u_1' = -\frac{x}{1+x^2} \Rightarrow \boxed{u_1 = -\frac{1}{2}\ln(1+x^2)}$ (3pts.)

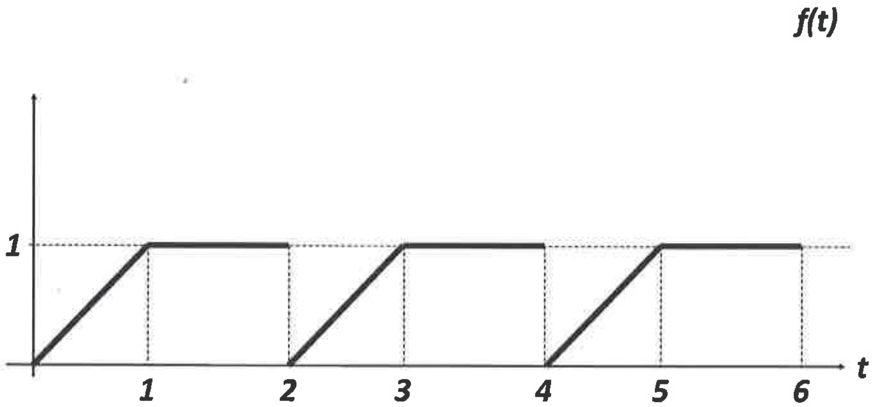
$$W_2 = \begin{vmatrix} e^x & 0 \\ e^x & \frac{e^x}{1+x^2} \end{vmatrix} = \frac{e^{2x}}{1+x^2}$$ (3pts.)

$u_2' = \frac{1}{1+x^2} \Rightarrow \boxed{u_2 = \arctan(x)}$ (3pts.)

$\boxed{y_p = -\frac{1}{2}e^x \ln(1+x^2) + xe^x \arctan(x)}$ (1pt.)

(1pt.) General Solution: $\boxed{y = y_c + y_p}$

5. Find the Laplace transform of the periodic function $f(t)$, pictured below:



Period $T=2$ (2pts.)

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-2s}}\,\mathcal{L}\Big\{\overbrace{t(1-u_1(t)) + (u_1(t)-u_2(t))}^{(5pts.)}\Big\}$$

$$= \frac{1}{1-e^{-2s}}\,\mathcal{L}\{t + (1-t)u_1(t) - u_2(t)\}$$

$$= \frac{1}{1-e^{-2s}}\left(\frac{1}{s^2} - e^{-s}\mathcal{L}\{t\} - \frac{e^{-2s}}{s}\right)$$

$$= \boxed{\frac{1}{1-e^{-2s}}\left(\underbrace{\frac{1}{s^2}}_{(1pt.)} - \underbrace{\frac{e^{-s}}{s^2}}_{(3pts.)} - \underbrace{\frac{e^{-2s}}{s}}_{(1pt.)}\right)}$$

$$= \boxed{\frac{1-e^{-s}-se^{-2s}}{s^2(1-e^{-2s})}}$$