Name: Solutions

September 23$^{\text{rd}}$, 2015.
Math 2552; Sections **L1** – **L4**.
Georgia Institute of Technology
Exam 1

I commit to uphold the ideals of honor and integrity by refusing to betray the trust bestowed upon me as a member of the Georgia Tech community. By signing my name below I pledge that I have neither given nor received help on this exam.

Pledged: ______________________________

Problem	Possible Score	Earned Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total	100	

Remember that you must SHOW YOUR WORK to receive credit!

Good luck!

1. Consider the autonomous equation:

$$\frac{dy}{dx} = (y+1)^2(4-y).$$

(a). Find the equilibrium solutions:

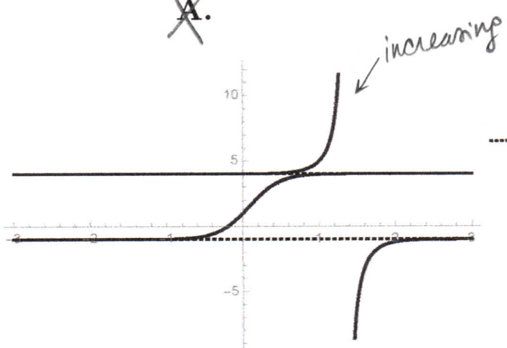
$y=-1;\ y=4$

(b). Draw the phase portrait.

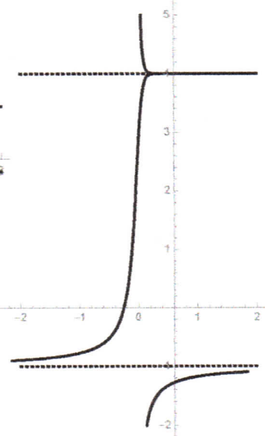
		-1		4	
$(y+1)^2$	$+\ +$	0	$+\ +\ +$	$+$	$+$
$(4-y)$	$+$	$+$	$+\ +$	0	$-\ -$
y'	$+$	0	$+$	0	$-$

(c). Determine which of the graphs below could be possible solutions to this equation (circle the correct one).

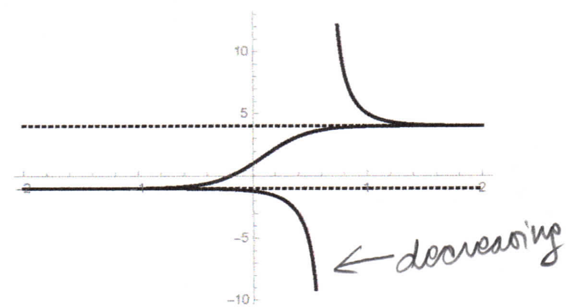
~~A.~~



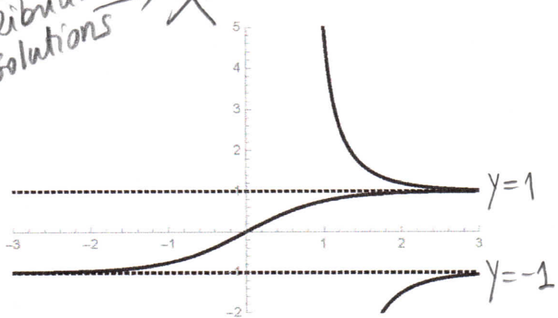
B.



~~C.~~



wrong equilibrium solutions → ~~D.~~



Grading Rubric:

(a). – 6 points (3pts./eq. sol)

(b). – 8 points –2 pts. for correct critical pts.

(c). – 6pts. –2pts. each orient.

2. Consider the differential equation:

$$y' + y\sin x = \sin x.$$

(a). If $y(x)$ is a solution to this equation and $y(\pi/2) = 2$, find $y(0)$.

Linear ; Integrating Factor : $\mu(x) = e^{\int \sin x\,dx} = e^{-\cos x}$

Multiply by $\mu(x)$: $\frac{d}{dx}\left(ye^{-\cos x}\right) = \sin x\, e^{-\cos x}$

$ye^{-\cos x} = \int \sin x\, e^{-\cos x}\,dx$

$= e^{-\cos x} + C$

$\boxed{y = 1 + Ce^{\cos x}}$

Solve IVP : $y = 2;\ x = \pi/2$

$2 = 1 + Ce^{0} \Rightarrow \boxed{C = 1} \Rightarrow y = 1 + e^{\cos x}$

$\Rightarrow y(0) = \boxed{1 + e}$

Alternate : Separable

$\frac{dy}{dx} = \sin x(1-y)$

$\frac{1}{1-y}\,dy = \sin x\,dx$

$-\ln|1-y| = -\cos x + C \Rightarrow \boxed{1 - y = Ce^{\cos x}} \Rightarrow \boxed{y = 1 + Ce^{-\cos x}}$

(b). If $y(x)$ is a solution to this equation and $y(\pi/2) = 1$, find $y(\pi^e)$.

Solve IVP : $y = 1;\ x = \pi/2 \Rightarrow 1 = 1 + C \Rightarrow \boxed{C = 0} \Rightarrow \boxed{y = 1} \Rightarrow y(\pi^e) = \boxed{1}$

Grading Rubric :

(a). – 16 points

Solve DE (linear)
- recognize linear (2pts.)
- integrating factor (3pts.)
- multiply by $\mu(x)$ (2pts.)
- integrate $\sin x\, e^{-\cos x}$ 3pts.
- general soln. (2pts.)

Solve IVP
- find $C = 1$ (2pts.)
- find $y(0) = 1 + e$ (2pts.)

(b). 4 points
- find $C = 0$ (2pts.)
- find $y(e^{\pi}) = 1$ (2pts.)

If solved DE as sep'ble:
- recognize sep'ble (2)
- put in sep'ble form (2)
- integrate both sides (2)
- antiderivatives (4)
- general sol. (2)

3. Find an explicit solution to the differential equation:

$$\frac{dx}{dy} = \frac{e^x}{2\sqrt{y}(e^{2x} - x)}.$$

Separable: $\displaystyle\int \frac{e^{2x}-x}{e^x}\,dx = \int \frac{1}{2\sqrt{y}}\,dy$

$$\int \frac{e^{2x}-x}{e^x}\,dx = \int e^x - xe^{-x}\,dx = e^x + \int x(e^{-x})'\,dx$$

$$= e^x + xe^{-x} - \int e^{-x}\,dx$$

$$= e^x + xe^{-x} + e^{-x} + c$$

$$\int \frac{1}{2\sqrt{y}}\,dy = \sqrt{y} + c$$

$$\Rightarrow \sqrt{y} = e^x + xe^{-x} + e^{-x} + c \Rightarrow \boxed{y = (e^x + xe^{-x} + e^{-x} + c)^2}$$

Grading Rubric:

- 2 pts. – recognize type
- 3 pts. – put in separable form
- 2 pts. – integrate both sides
- 8 pts. – integration of $\frac{e^{2x}-x}{e^x}$
  - 2 pts. – separate into $e^x - \frac{x}{e^x}$
  - 2 pts. – integrate e^x
  - 4 pts. – integrate $-\frac{x}{e^x}$ (by parts).
- 3 pts. – integration of $\frac{1}{2\sqrt{y}}$
- 2 pts. – final answer

4. Find the value of k for which the following differential equation is exact:

$$\left(ye^{xy} + y^2 - \frac{y}{x^2}\right)\, dx + \left(xe^{xy} + kxy + \frac{1}{x} + 2y\right)\, dy = 0,$$

and solve the equation for this value of k (give an *implicit* answer).

Find k: $M_y = e^{xy} + xye^{xy} + 2y - \frac{1}{x^2}$

$N_x = e^{xy} + xye^{xy} + ky - \frac{1}{x^2}$

$\Rightarrow k=2$

Solve ODE: $\left(ye^{xy} + y^2 - \frac{y}{x^2}\right) dx + \left(xe^{xy} + 2xy + \frac{1}{x} + 2y\right) dy = 0$

Find potential: $\frac{\partial f}{\partial x} = ye^{xy} + y^2 - \frac{y}{x^2} \Rightarrow f(x,y) = e^{xy} + xy^2 + \frac{y}{x} + g(y)$

$\Rightarrow \frac{\partial f}{\partial y} = xe^{xy} + 2xy + \frac{1}{x} + g'(y)$

$\Rightarrow g'(y) = 2y \Rightarrow g(y) = y^2$

$f(x,y) = e^{xy} + xy^2 + \frac{y}{x} + y^2$

Solutions: $\boxed{e^{xy} + xy^2 + \frac{y}{x} + y^2 = c}$

Grading Rubric:

- Find K: 6 points
  - partial M_y (2pts)
  - partial N_x (2pts)
  - $K=2$ (2pts)
- Find potential: 12 points
  - intent (2pts)
  - 1st integration: (6pts)
  - partial deriv. of $f(x,y)$ obtained @ first integration: (2pts)
  - 2nd integration: (2pts)
- Final answer: 2 points

5. Solve the differential equation:

$$y' + 4xy = 4x\sqrt{y},$$

and give an *explicit* solution.

Bernoulli with $\alpha = 1/2$

$\Rightarrow 1-\alpha = 1/2 \Rightarrow$ Substitution: $u = y^{1/2}$

$\Rightarrow$ Eqn. becomes

$$\frac{du}{dx} + \frac{1}{2} \cdot 4x \cdot u = \frac{1}{2} \cdot 4x$$

$$\frac{du}{dx} + 2xu = 2x$$

Integrating factor: $e^{\int 2x\,dx} = e^{x^2}$

Multiply by $\mu(x)$: $\frac{d}{dx}(ue^{x^2}) = 2xe^{x^2}$

$\Rightarrow ue^{x^2} = e^{x^2} + C$

$\Rightarrow u = 1 + Ce^{-x^2}$

$\Rightarrow y^{1/2} = 1 + Ce^{-x^2}$

$$\boxed{y = (1+Ce^{-x^2})^2}$$

Grading Rubric:

- 3 pts. – recognizing Bernoulli & α
- 3 pts. – correct substitution ($u = \sqrt{y}$)
- 3 pts. – put eqn. in linear form ($u' + 2xu = 2x$)
- 3 pts. – integrating factor ($\mu(x) = e^{x^2}$)
- 2 pts. – multiply by $\mu(x)$
- 3 pts. – integrate $2xe^{x^2}$
- 1 pt. – solve for u
- 2 pts. – final answer for y