Name: Solutions

September 23$^{\text{rd}}$, 2015.
Math 2552; Sections **F1** – **F4**.
Georgia Institute of Technology
Exam 1

I commit to uphold the ideals of honor and integrity by refusing to betray the trust bestowed upon me as a member of the Georgia Tech community. By signing my name below I pledge that I have neither given nor received help on this exam.

Pledged: ______________________

Problem	Possible Score	Earned Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total	100	

Remember that you must SHOW YOUR WORK to receive credit!

Good luck!

1. Consider the autonomous equation:

$$\frac{dy}{dx} = y^2(y-3).$$

(a). Find the equilibrium solutions:

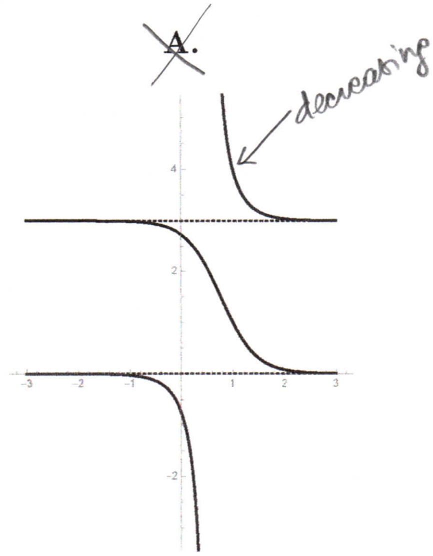
$y=0;\ y=3$

(b). Draw the phase portrait.

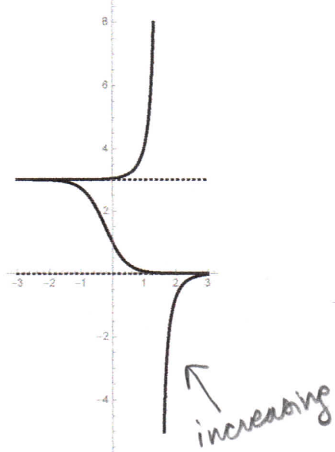
y	0 $\quad$ 3
y^2	$+ + 0 + + + +$
$y-3$	$- - - - - - 0 + +$
y'	$- \ 0 \ - \ 0 \ +$

(c). Determine which of the graphs below could be possible solutions to this equation (circle the correct one).

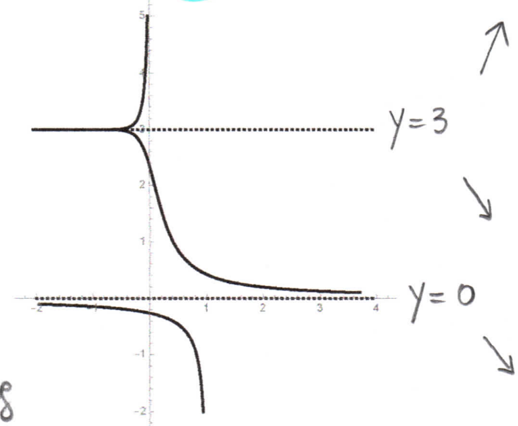
~~A.~~



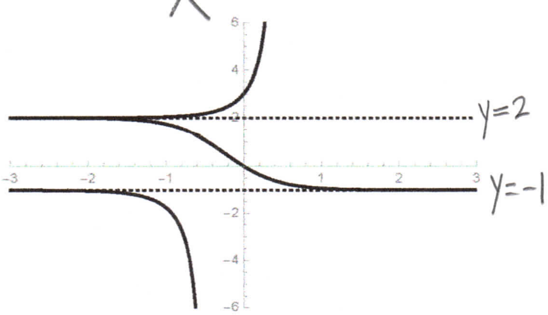
~~B.~~



C.



~~D.~~ ← wrong equilibrium solutions



Grading Rubric

(a) – 6 points (3pt. for each eq. sol.)

(b) – 8 points – 2 pts. for correct critical pts.
– 2pts. each orientation

(c) – 6 points

2. Consider the differential equation:

$$\frac{dy}{dx} = \frac{y}{x \ln x}.$$

(a). If $y(x)$ is a solution to this equation and $y(e) = 1$, find $y(e^2)$.

Separable : $\frac{1}{y}dy = \frac{1}{x \ln x}dx$

$\int \frac{1}{y}dy = \int \frac{1}{x \ln x}dx$

$\ln|y| = \ln|\ln x| + c$

$e^{\ln|y|} = e^{\ln|\ln x| + c}$

$|y| = e^c |\ln x|$

$y = \pm e^c \ln x$

$y = c \ln x$

$\boxed{y = c \ln x}$

Solve IVP : $x = e; y = 1$

$1 = c\ln(e) \Rightarrow \boxed{c = 1}$

$\Rightarrow y = \ln x$

$\Rightarrow y(e^2) = \ln(e^2) = \boxed{2}$

(b). If $y(x)$ is a solution to this equation and $y(e) = 0$, find $y(e^\pi)$.

Solve IVP : $x = e; y = 0$

$0 = c\ln(e) \Rightarrow \boxed{c = 0}$

$\Rightarrow \boxed{y = 0} \Rightarrow \boxed{y(e^\pi) = 0}$

Grading Rubric

(a). – 16 points

Solve DE:
- bring to separable form: (2pts.)
- integrate both sides: (2pts.)
- antiderivatives: (6pts.)
- gen. solution: (2pts.)

Solve IVP:
- find $c = 1$ (2pts.)
- find $y(e^2) = 2$ (2pts.)

(b). – 4 points
- find $c = 0$ (2pts.)
- find $y(e^\pi) = 0$ (2pts.)

3. Find the value of k for which the following differential equation is exact:

$$(xy^2 + xe^y)\,dx + (x^2y + kx^2e^y + 3y^2)\,dy = 0,$$

and solve the equation for this value of k (give an *implicit* solution).

Find k:

$$\left.\begin{array}{l} \frac{\partial M}{\partial y} = 2xy + xe^y \\ \frac{\partial N}{\partial x} = 2xy + 2kxe^y \end{array}\right\} \Rightarrow \boxed{k = 1/2}$$

Solve ODE:

$$(xy^2 + xe^y)dx + (x^2y + \tfrac{1}{2}x^2e^y + 3y^2)dy = 0$$

Potential:

$$\frac{\partial f}{\partial x} = xy^2 + xe^y \Rightarrow f(x,y) = \tfrac{1}{2}x^2y^2 + \tfrac{1}{2}x^2e^y + g(y)$$

$$\left.\begin{array}{l} \Rightarrow \frac{\partial f}{\partial y} = x^2y + \frac{1}{2}x^2e^y + g'(y) \\ \quad = x^2y + \frac{1}{2}x^2e^y + 3y^2 \end{array}\right\} \Rightarrow$$

$$\Rightarrow 3y^2 = g'(y) \Rightarrow g(y) = y^3$$

$$f(x,y) = \tfrac{1}{2}x^2y^2 + \tfrac{1}{2}x^2e^y + y^3$$

Solutions:

$$\boxed{\tfrac{1}{2}x^2y^2 + \tfrac{1}{2}x^2e^y + y^3 = C}$$

Grading Rubric:

- Finding k: 6 points
  - partial M_y (2 pts.)
  - partial N_x (2 pts.)
  - $k = 1/2$ (2 pts.)
- Finding potential: 12 points
  - intent (2 pts.)
  - 1st integration: (6 pts.)
  - partial derivative of $f(x,y)$ obtained after 1st integration: (2 pts.)
  - 2nd integration: (2 pts.)
- Final answer: 2 points.

4. Find an explicit solution to the initial value problem

$$x\,dy = (x\cos x - y)\,dx; \quad y(\pi/2) = 1,$$

and state the largest interval on which your solution is valid.

$$x\frac{dy}{dx} = x\cos x - y$$

$$x\frac{dy}{dx} + y = x\cos x \qquad (\underline{\text{linear}})$$

Standard Form:

$$\frac{dy}{dx} + \frac{1}{x}y = \cos x$$

$$p(x) = \frac{1}{x}$$

$$\int p(x) = \ln|x|$$

$$\boxed{\mu(x) = x}$$ (can work on $(0,\infty)$ b/c of initial condition)

Multiply by $\mu(x)$: $\frac{d}{dx}(xy) = x\cos x$

$$xy = \int x\cos x\,dx$$
$$= \int x(\sin x)'\,dx$$
$$= x\sin x - \int \sin x\,dx$$
$$= x\sin x + \cos x + C$$

$$\boxed{y = \sin x + \frac{1}{x}\cos x + \frac{C}{x}}$$

Grading Rubric:

- 3pts. – recognizing the eqn. is Linear
- 4pts. – bringing it to the form $\frac{d}{dx}(xy) = x\cos x$ (either by just recognizing this or by integrating factor)
- 5pts. – integrate $x\cos x$
- 3pts. – find $c=0$
- 3pts. – explicit solution
- 2pts. – interval $(0,\infty)$.

If Using Exactness:

- 2pts. – recognize exact
- 1pt. – intent to find potential
- 1pt. – first integration
- 2pts. – partial of first int.
- 5pts. – integrate $x\cos x$
- 1pt. – potential
- 3pts. – find $c=0$
- 3pts. – ~~explicit sol.~~
- 2pts. – interval $(0,\infty)$

Solve IVP: $x = \pi/2;\ y = 1$

$$1 = 1 + \frac{C}{\pi/2} \Rightarrow \boxed{C=0} \Rightarrow \boxed{y = \sin x + \frac{1}{x}\cos x}, \ \boxed{x\in(0,\infty)}$$

or: Exact: $(x\cos x - y)dx - x\,dy = 0$

$$M_y = -1 \qquad N_x = -1$$

Potential: $\frac{\partial f}{\partial x} = x\cos x - y$

$$\Rightarrow f(x,y) = x\sin x + \cos x - xy + g(y)$$

$$\left.\begin{aligned}\frac{\partial f}{\partial y} &= -x + g'(y)\\ &= -x\end{aligned}\right\} \Rightarrow g(y) = 0$$

$$f(x,y) = x\sin x + \cos x - xy$$

Solution:

$$x\sin x + \cos x - xy = C$$

$$\boxed{y = \sin x + \frac{1}{x}\cos x + \frac{C}{x}}$$

5. Find an *explicit* solution to the initial value problem:

$$\frac{dy}{dx} = \frac{y}{x} \ln \frac{y}{x}; \quad y(1) = 1,$$

and state the largest interval where your solution is valid.

Homogeneous of degree 0 : $\boxed{u = \frac{y}{x}}$ $\Rightarrow$ $\boxed{\frac{dy}{dx} = u + x\frac{du}{dx}}$

$$u + x\frac{du}{dx} = u \ln u$$

$$x\frac{du}{dx} = u(\ln u - 1)$$

$$\int \frac{1}{u(\ln u - 1)}\,du = \int \frac{1}{x}\,du$$

$$\ln|\ln u - 1| = \ln|x| + c$$

$$\ln u - 1 = cx$$

$$\ln u = cx + 1$$

$$u = e^{cx+1}$$

$$\frac{y}{x} = e^{cx+1} \Rightarrow y = xe^{cx+1}$$

Solve IVP : $y = 1;\ x = 1 \Rightarrow 1 = e^{c+1} \Rightarrow c + 1 = 0 \Rightarrow c = -1$

$\Rightarrow \boxed{y = xe^{1-x}}$ interval : $\boxed{(0, \infty)}$

(look at original ODE: $x \neq 0$ and IVP forces $(0, \infty)$).

Grading Rubric:

2 pts. - recognize homogeneous
2 pts. - correct substitution ($u = y/x$ or $y = ux$)
2 pts. - correct dy/dx (or dy)
2 pts. - put in separable form & integrate both sides
3 pts. - antiderivative $\ln|\ln u - 1|$
1 pt. - antiderivative $\ln|x|$
3 pts. - bring to explicit form
2 pts. - find c
1 pt. - final answer for y
2 pts. - interval

F4 - #5. Another method:

$$\frac{dy}{dx} = \frac{y}{x} \ln \frac{y}{x}$$

$$\Leftrightarrow \frac{1}{y}\frac{dy}{dx} = \frac{1}{x}(\ln y - \ln x).$$

let $u = \ln y$

$$\Rightarrow \frac{du}{dx} = \frac{1}{x}(u - \ln x) \Rightarrow \frac{du}{dx} - \frac{1}{x}u = -\frac{\ln x}{x}$$

$$\Rightarrow \frac{d}{dx}\left(\frac{1}{x}u\right) = -\frac{\ln x}{x^2}$$

$$\Rightarrow \frac{1}{x}u = -\int \frac{\ln x}{x^2}\,dx$$

$$\int \frac{\ln x}{x^2}\,dx \overset{\text{let } v = \ln x}{=\!=\!=\!=} \int \frac{v}{e^v}\,dv$$

$$= \int v e^{-v}\,dv = -ve^{-v} - e^{-v} + C$$

$$= -\ln x \cdot \frac{1}{x} - \frac{1}{x} + C$$

$$\Rightarrow \frac{1}{x}u = \frac{\ln x}{x} + \frac{1}{x} + C$$

$$\Rightarrow u = \ln x + 1 + Cx$$

$$\Rightarrow y = e^{\ln x + 1 + Cx} = x e^{Cx+1}$$