NAME: Solutions

Georgia Tech, Fall 2015
Math 2552 (Sections F1 – F4)

Quiz 3

(10 pts.) **1.** Solve the ODE below. Give an *explicit* solution.

$$x\frac{dy}{dx} - 2y = x^4 e^x.$$

(10 pts.) **2.** Solve the ODE below. Give an *implicit* solution in the form $x = f(x, y)$.

$$(y^2 - xy)\,dx + x^2\,dy = 0.$$

1 Linear (+1) pt. – recognizing type

Standard form: (+1) pt. – put in standard form

$$\frac{dy}{dx} - \frac{2}{x}y = x^3 e^x$$

Integrating factor: (+3) pts. – finding int. fact.

$$\int p(x) = \int -\frac{2}{x}\,dx = -2\ln|x|$$

$$\mu(x) = e^{-2\ln|x|} = \frac{1}{|x|^2} = \frac{1}{x^2}$$

Multiply by Int. factor: (+1) pt. – multiply by $\mu(x)$, new form

$$\frac{d}{dx}\left(\frac{1}{x^2}y\right) = xe^x$$

Solve for y: (+3) pts. – solve for y

$$\frac{1}{x^2}y = \int xe^x\,dx$$
$$= \int x(e^x)'\,dx$$
$$= xe^x - \int e^x\,dx$$
$$= xe^x - e^x + c$$

$$\boxed{y = x^2(xe^x - e^x + c)}$$

(+1) pt. – final answer

OR: $(1 - x/y)\frac{dx}{dy} + (x/y)^2 = 0$; $\boxed{u = x/y}$ => $\frac{dx}{dy} = u + y\frac{du}{dy}$

$$(1-u)\left(u + y\frac{du}{dy}\right) + u^2 = 0$$
$$u = y(u-1)\frac{du}{dy}$$
$$\frac{1}{y}dy = \left(1 - \frac{1}{u}\right)du \ldots\ldots$$

2 Homogeneous (degree 2)

Substitution: $\boxed{y = ux}$; $dy = u\,dx + x\,du$

$$(u^2x^2 - ux^2)\,dx + x^2(u\,dx + x\,du) = 0$$
$$(u^2 - u)\,dx + (u\,dx + x\,du) = 0$$
$$u^2\,dx + x\,du = 0$$
$$u^2\,dx = -x\,du$$
$$\int \frac{1}{x}\,dx = \int -\frac{1}{u^2}\,du$$
$$\ln|x| = \frac{1}{u} + c$$
$$\ln|x| = \frac{x}{y} + c$$
$$|x| = e^{x/y + c}$$
$$|x| = ce^{x/y}$$
$$x = \pm ce^{x/y}$$

$$\boxed{x = ce^{x/y}}$$

Either way:
(+1) pt. – recognize type
(+2) pt. – substitution
(+2) pt. – bring to separable form
(+3) pt. – solve separable eqn.
(+2) pt. – bring to final answer form

OR: $(y/x)^2 - y/x + \frac{dy}{dx} = 0$

Substitution: $\boxed{u = y/x}$ => $\frac{dy}{dx} = u + x\frac{du}{dx}$

$$u^2 - \cancel{u} + \cancel{u} + x\frac{du}{dx} = 0 \Rightarrow u^2\,dx = -x\,du$$
$$\vdots$$

OR: $\boxed{x = uy}$ => $dx = u\,dy + y\,du$

$$(y^2 - uy^2)(u\,dy + y\,du) + u^2y^2\,dy = 0$$
$$(1-u)(u\,dy + y\,du) + u^2\,dy = 0$$
$$u\,dy = (u-1)y\,du$$
$$\frac{1}{y}dy = \left(1 - \frac{1}{u}\right)du$$
$$\ln|y| = u - \ln|u| + C \Rightarrow \ln|x| = \frac{x}{y} + C \ldots$$

NAME: ..Solutions..........................

Georgia Tech, Fall 2015
Math 2552 (Sections [redacted])
L1 – L4

Quiz 3

(10 pts.) **1.** Solve the ODE below. Give an *implicit* solution in the form $x = f(x, y)$.

$$xy\,dy = (x^2 + y^2)\,dx.$$

(10 pts.) **2.** Solve the ODE below. Give an *implicit* solution.

$$(y^2x - 3) + (yx^2 + 4)\frac{dy}{dx} = 0.$$

2 $(y^2x-3)dx + (yx^2+4)dy = 0$

$\frac{\partial M}{\partial y} = 2xy;\ \frac{\partial N}{\partial x} = 2xy \Rightarrow$ exact (+2) pts.

Find potential: (+6) pts.

$\frac{\partial f}{\partial x} = y^2x - 3$

$\Rightarrow f(x,y) = \frac{1}{2}x^2y^2 - 3x + g(y)$

$\left.\begin{aligned}\Rightarrow \frac{\partial f}{\partial y} &= x^2y + g'(y)\\ &= x^2y + 4\end{aligned}\right\} \Rightarrow g'(y) = 4,\quad g(y) = 4y$

$f(x,y) = \frac{1}{2}x^2y^2 - 3x + 4y$

Solutions:

$\boxed{\frac{1}{2}x^2y^2 - 3x + 4y = c}$ (+2) pts.

1 Homogeneous (degree 2)

Substitution: $\boxed{y = ux} \Rightarrow dy = u\,dx + x\,du$

$ux^2(u\,dx + x\,du) = (x^2 + u^2x^2)dx$

$u^2dx + ux\,du = (1+u^2)dx$

$ux\,du = dx$

$u\,du = \frac{1}{x}dx$

$\ln|x| = \frac{1}{2}u^2 + C$

$\ln|x| = \frac{1}{2}y^2/x^2 + C$

$|x| = Ce^{1/2\,y^2/x^2} \Rightarrow x = \pm Ce^{1/2\,y^2/x^2}$

$\boxed{x = Ce^{1/2\,y^2/x^2}}$

OR: $\frac{y}{x}\frac{dy}{dx} = 1 + (y/x)^2$

$u = y/x \Rightarrow \frac{dy}{dx} = u + x\frac{du}{dx}$

$u\left(u + x\frac{du}{dx}\right) = 1 + u^2$

$ux\frac{du}{dx} = 1 \Rightarrow ux\,du = dx$ .....

OR: $\boxed{x = uy}$ $dx = u\,dy + y\,du$

$uy^2dy = (u^2+1)y^2(u\,dy + y\,du)$

$(u - u^3 - u)dy = y(u^2+1)du$

$\frac{1}{y}dy = -\frac{u^2+1}{u^3}du$

$\ln|y| = -\ln|u| + \frac{1}{2}\frac{1}{u^2} + C$

$\ln|x| = \frac{1}{2}\frac{y^2}{x^2} + C$

$\boxed{x = Ce^{1/2\,y^2/x^2}}$

OR: $\frac{x}{y} = \left(\left(\frac{x}{y}\right)^2 + 1\right)\frac{dx}{dy}$

$\boxed{u = x/y}$ $\frac{dx}{dy} = u + y\frac{du}{dy}$

$u = (u^2+1)\left(u + y\frac{du}{dy}\right)$

$-u^3 = (u^2+1)y\frac{du}{dy}$

⋮

Either way:

(+1) pt. – recognize type

(+2) pts. – substitution

(+2) pts. – bring to separable form

(+3) pts. – solve separable eqn.

(+2) pts. – bring to final answer form