Classifications of DEs :

① By Type

Ordinary Differential Equations (ODE) (a single independent variable)

Ex): $\frac{du}{dx} - \frac{d^2v}{dx^2} = x$; $\frac{d^3y}{dt^3} - 2\frac{dy}{dt} + 3t = 0$

Partial Differential Equations (PDE) (2 or more independent variables)

Ex): $\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}$; $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial u}{\partial y}$

② By Order:

The order of a DE is the order of the highest-order derivative.

Ex): $\frac{d^2y}{dt^2} + 5\left(\frac{dy}{dt}\right)^3 - 4y = e^x$ Second Order ODE

$4\frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial x \partial y} = 0$ Fourth Order PDE

③ By Linearity :

An n^{th} order ODE is linear if it can be written in the form:

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \dots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

Characteristics:

- The dependent variable and all of its derivatives are raised to the power 1.
- Each coefficient $(a_n, a_{n-1}, \dots, a_0, g)$ depends only on the independent variable.

Ex): $x\frac{dy}{dx} + y = 0$ Linear, 1^{st} order

Ex): $x^3\frac{d^2y}{dx^2} + \sin(x)\frac{dy}{dx} = e^x$ Linear 2^{nd} order

Ex): $e^{\cos x}y^{(6)} + 7^x y' + xy = 0$ Linear 7^{th} order

Ex): $yy'' - 2y' = x$ Non-linear, 2^{nd} order.

Ex): $y^{(3)} + (y')^2 + y = 0$ Non-linear, 3^{rd} order.

Ex): $y' = \frac{y}{y-x}$ Non-linear (in y), 1^{st} order   [Reciprocal: $x' = \frac{y-x}{y} = 1 - \frac{1}{y}x$ is linear in x]

Ex): $y' = \frac{xy}{y-x}$ Non-linear (in y), 1^{st} order   [Reciprocal: $x' = \frac{y-x}{xy}$
$x' = \frac{1}{x} - \frac{1}{y}$
non-linear in x]

• Separable Equations : $\boxed{\dfrac{dy}{dx} = \dfrac{g(x)}{h(y)}}$

Method : $\int h(y)\,dy = \int g(x)\,dx$

Integrate & solve for y (or get an implicit solution)

Ex] : $(1+x)dy - y\,dx = 0$

$$(1+x)\,dy = y\,dx$$
$$\frac{1}{y}\,dy = \frac{1}{1+x}\,dx$$
$$\ln|y| = \ln|1+x| + C$$
$$|y| = c\,|1+x| \Rightarrow y = \pm c(1+x)$$
$$\boxed{y = c\,(1+x)}$$

• **Autonomous Equations** : $\boxed{\dfrac{dy}{dx} = f(y)}$

- * Separable : If $f(y) \neq 0$: $\frac{1}{f(y)}dy = dx$
- * Critical Points : c s.t. $f(c) = 0$
- * Equilibrium Solutions : $y = c$ where c = critical pt.
- * Phase portraits

• Linear Equations : $\boxed{\dfrac{dy}{dx} + p(x)\,y = g(x)}$ (Standard Form)

Method : Find Integrating Factor : $\mu(x) = e^{\int p(x)dx}$

(from Standard Form)

Multiply eqn. by $\mu(x)$

$\Rightarrow$ Eqn. becomes $\dfrac{d}{dx}\left(y\,\mu(x)\right) = g(x)\mu(x)$

Integrate : $y\,\mu(x) = \int g(x)\mu(x)\,dx$

Solve for y.

Ex]: $\dfrac{dy}{dx} + 3y = x$

$$\mu(x) = e^{\int 3dx} = e^{3x}$$
$$e^{3x}y' + 3e^{3x}y = xe^{3x}$$
$$\frac{d}{dx}\left(e^{3x}y\right) = xe^{3x}$$
$$e^{3x}y = \int xe^{3x} = \frac{1}{3}\int x(e^{3x})'dx$$
$$= \frac{1}{3}\left(xe^{3x} - \int e^{3x}dx\right)$$
$$= \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + C$$
$$\boxed{y = \frac{1}{3}x - \frac{1}{9} + Ce^{-3x}}$$

- **Exact Equations:** $\boxed{M(x,y)\,dx+N(x,y)\,dy=0\ ;\ \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}}$

Method: Find potential: $f(x,y)$ s.t. $\frac{\partial f}{\partial x}=M\ ;\ \frac{\partial f}{\partial y}=N$

$\Rightarrow$ Eqn. becomes $\underbrace{\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\frac{dy}{dx}}=0$

Chain Rule: $\frac{df}{dx}=0 \Rightarrow \boxed{f(x,y)=c}$ Solution

Ex): $2xy\,dx+(x^2-1)\,dy=0$

$\frac{\partial M}{\partial y}=2x;\ \frac{\partial N}{\partial x}=2x \Rightarrow$ exact

Potential: $\frac{\partial f}{\partial x}=2xy \Rightarrow f(x,y)=x^2y+g(y)$

$$\Rightarrow \left.\begin{aligned}\frac{\partial f}{\partial y}&=x^2+g'(y)\\ &=x^2-1\end{aligned}\right\}\Rightarrow g'(y)=-1 \Rightarrow g(y)=-y$$

$$\Rightarrow f(x,y)=x^2y-y \Rightarrow (x^2-1)y=c \Rightarrow \boxed{y=\frac{c}{x^2-1}}$$

- **Homogenous Eqns.:** $M(x,y)\,dx+N(x,y)\,dy=0$; M,N homogeneous of same degree:

$$M(\alpha x,\alpha y)=\alpha^k M(x,y)$$
$$N(\alpha x,\alpha y)=\alpha^k N(x,y)$$

Method: Either one of the substitutions $\underbrace{y=ux}_{dy=u\,dx+x\,du}$ or $\underbrace{x=uy}_{dx=u\,dy+y\,du}$ turns the eqn. separable.

OR: write eqn. as $\frac{dy}{dx}=F\left(\frac{y}{x}\right)$ & make the substitution $\underbrace{u=\frac{y}{x}}_{\frac{dy}{dx}=u+x\frac{du}{dx}}$ $\Rightarrow$ separable

OR: write eqn. as $\frac{dx}{dy}=F\left(\frac{x}{y}\right)$ & make the substitution $\underbrace{u=\frac{x}{y}}_{\frac{dx}{dy}=u+y\frac{du}{dy}}$ $\Rightarrow$ separable

Ex]: $(x^2+y^2)\,dx+(x^2-xy)\,dy=0$ (homogeneous of degree 2)

$\boxed{y=ux}$ $\Rightarrow$ $(x^2+u^2x^2)dx+(x^2-x^2u)(u\,dx+x\,du)=0$

$dy=u\,dx+x\,du$

$$(1+u^2)dx+(1-u)(u\,dx+x\,du)=0$$

$$(1+u)\,dx=(u-1)x\,du$$

$$\frac{1}{x}dx=\frac{u-1}{u+1}du \Rightarrow \ln|x|=\int\frac{u+1-2}{u+1}du=\int 1-\frac{2}{u+1}\,du$$

$$=u-2\ln|u+1|+C$$

OR: Divide by x^2 and write eqn. as:

$$(1+(y/x)^2)+(1-y/x)\frac{dy}{dx}=0$$

$\boxed{u=y/x}$ $\Rightarrow$ $(1+u^2)+(1-u)\left(u+x\frac{du}{dx}\right)=0$

$\frac{dy}{dx}=u+x\frac{du}{dx}$

$$1+u=(u-1)x\frac{du}{dx}$$

$$(1+u)dx=(u-1)x\,du$$

$\vdots$

$$\Rightarrow \ln|x|+2\ln|u+1|-u=C$$

$$\ln|x(u+1)^2|-u=C$$

$$\ln\left|x\left(\frac{y}{x}+1\right)^2\right|-\frac{y}{x}=\ln|c|$$

$$\ln\left|\frac{(x+y)^2}{cx}\right|=+\frac{y}{x} \Rightarrow \frac{(x+y)^2}{cx}=e^{y/x}$$

$$\boxed{(x+y)^2=cxe^{y/x}}$$

- **Bernoulli Equations**: $\boxed{\frac{dy}{dx}+P(x)y=Q(x)y^{\alpha}}$ $\alpha\in\mathbb{R}$

Method: If $\alpha=0,\alpha=1 \Rightarrow$ linear

If $\boxed{\alpha\neq 1}$: substitution $\boxed{u=y^{1-\alpha}}$ $\Rightarrow \frac{du}{dx}=(1-\alpha)y^{-\alpha}\frac{dy}{dx} \Rightarrow y^{-\alpha}\frac{dy}{dx}=\frac{1}{1-\alpha}\frac{du}{dx}$

Divide eqn. by y^{α}: $\underbrace{y^{-\alpha}\frac{dy}{dx}}+P(x)\underbrace{y^{1-\alpha}}=Q(x)$

$$\frac{1}{1-\alpha}\frac{du}{dx}+P(x)u=Q(x) \Rightarrow \boxed{\frac{du}{dx}+(1-\alpha)P(x)u=(1-\alpha)Q(x)}$$

(linear)

Solve for u,
use u to find y.

Ex]: $y'=\underset{P(x)}{5}y+\underset{Q(x)}{e^{-2x}}y^{-2}$; $y'-5y=e^{-2x}y^{-2}$ Bernoulli w/ $\alpha=-2$

$1-\alpha=3 \Rightarrow \boxed{u=y^3}$

$$\Rightarrow u'+3(-5)u=3\cdot e^{-2x}$$

$$\Rightarrow u'-15u=3e^{-2x}$$

$$\mu(x)=e^{-15x} \Rightarrow \frac{d}{dx}\left(ue^{-15x}\right)=3e^{-17x}$$

$$\Rightarrow ue^{-15x}=-\frac{3}{17}e^{-17x}+c \Rightarrow u=\boxed{y^3=-\frac{3}{17}e^{-2x}+ce^{15x}}$$

FIRST ORDER ODEs

$\frac{dy}{dx} = f(x,y)$

Autonomous: $\frac{dy}{dx} = f(y)$

Separable: $\frac{dy}{dx} = \frac{g(x)}{h(y)}$

Method: Separate, Integrate, Solve

$\int h(y)\,dy = \int g(x)\,dx$

Exact: $M(x,y)dx + N(x,y)dy = 0$; $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Method: Find potential: $f(x,y)$ s.t. $\frac{\partial f}{\partial x} = M$; $\frac{\partial f}{\partial y} = N$

=> Solution: $f(x,y) = c$

Linear: $\frac{dy}{dx} + p(x)y = g(x)$

Method: Find integrating factor:

$\mu(x) = e^{\int p(x)dx}$

Multiply eqn. by $\mu(x)$ & it becomes:

$\frac{d}{dx}\left(y\mu(x)\right) = \mu(x)g(x)$

Integrate: $y\mu(x) = \int \mu(x)g(x)dx$

Solve for y.

Homogeneous: $M(x,y)dx + N(x,y)dy = 0$

M, N homogeneous of same degree:

$M(\alpha x, \alpha y) = \alpha^k M(x,y)$

$N(\alpha x, \alpha y) = \alpha^k N(x,y)$

Method: Substitution: Either one of the substitutions:

$\boxed{y = ux}$ $\rightsquigarrow$ $\boxed{dy = x\,du + u\,dx}$

$\boxed{u = \frac{y}{x}}$

turns it into a separable ODE

Bernoulli: $\frac{dy}{dx} + p(x)y = Q(x)y^{\alpha}$; $\alpha \in \mathbb{R}$

Method: $\alpha = 0$ or $\alpha = 1$ => linear

$\boxed{\alpha \neq 1}$: Substitution $\boxed{u = y^{1-\alpha}}$

=> Eqn. becomes: $\boxed{\frac{du}{dx} + (1-\alpha)P(x)u = (1-\alpha)Q(x)}$

(linear)

HIGHER ORDER LINEAR ODEs

n^{th} order linear ODE: $a_n(x)\frac{d^ny}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\ldots+a_1(x)\frac{dy}{dx}+a_0(x)y=g(x)$

IVP: The dependent variable y and its derivatives are specified at the same value x_0 of x (indp. var.):

$y(x_0)=b_0$; $y'(x_0)=b_1$; …; $y^{(n-1)}(x_0)=b_{n-1}$

for some numbers $b_0, b_1, \ldots, b_{n-1}$

BVP: The dependent variable y and its derivatives are specified at different values of x (indp. var.)

Ex: For a 2nd order linear eqn:

$$a_2(x)y''+a_1(x)y'+a_0(x)y=g(x)$$

one can have the following possibilities for the boundary value conditions:

- $y(a)=b_0$; $y(b)=b_1$
- $y'(a)=b_0$; $y(b)=b_1$
- $y(a)=b_0$; $y'(b)=b_1$
- $y'(a)=b_0$; $y'(b)=b_1$

Homogeneous if $g=0$
(different meaning of the word "homogeneous")

Non-Homogeneous if $g\neq 0$

Ex: $y'''+2y''-e^xy=x^2$ (non-homogeneous)

$y^{(4)}-xe^xy''+y=0$ (homogeneous)

- To solve a non-homogeneous linear ODE, we must first solve the homogeneous one.

Existence & Uniqueness for IVPs:

Let an n^{th} order linear ODE: $a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_0 y = g$; $y(x_0) = b_0, \ldots, y^{(n-1)}(x_0) = b_{n-1}$

Suppose that an interval $I \subset \mathbb{R}$ satisfies:

- $a_n(x), a_{n-1}(x), \ldots, a_0(x)$ and $g(x)$ are continuous on I.
- $a_n(x) \neq 0$ for all $x \in I$.

Then, for every $x_0 \in I$, there exists a unique solution to the IVP.

Ex]: $y = 3e^{2x} + e^{-2x} - 3x$ is a solution to: $y'' - 4y = 12x$; $y(0) = 4$; $y'(0) = 1$

By the Theorem above, it is the unique solution.
$a_2(x) = 1$ continuous, never 0
$a_1(x) = 0$ continuous
$a_0(x) = -4$ continuous
$g(x) = 12x$ continuous.

Ex]: $y = cx^2 - x$ is a solution to $x^2 y'' - 2xy' + 2y = 0$; $y(0) = 0$; $y'(0) = -1$ for any value of c

(an IVP with ∞-many solutions).

(the assumptions of the Thm. are not met: $a_2(x) = x^2$; $a_2(0) = 0$.)

- Even when the assumptions of the Thm. above are satisfied, a BVP can have
  - * several solutions (∞-many)
  - * one solution
  - * no solutions.

Ex]: $y'' + 16y = 0$

$y = c_1 \cos(4x) + c_2 \sin(4x)$ – general solution –

Impose different boundary conditions:

- **BVP 1**: $y(0) = 0$; $y(\pi/2) = 0$ $\Rightarrow$ ∞-many solutions ($y = c_2 \sin(4x)$ is a sol. for all c_2)
- **BVP 2**: $y(0) = 0$; $y(\pi/8) = 0$ $\Rightarrow$ one solution ($y = 0$)
- **BVP 3**: $y(0) = 0$; $y(\pi/2) = 1$ $\Rightarrow$ no solutions

Linear Independence

Def.: A set of n functions $f_1(x), \dots, f_n(x)$ are called <u>linearly dependent</u> on some interval $I \subset \mathbb{R}$ if there exist constants $c_1, c_2, \dots, c_n$ (not all zero) such that:

$$\boxed{c_1 f_1(x) + c_2 f_2(x) + \dots + c_n f_n(x) = 0 \ , \ \forall x \in I}$$

Otherwise, they are called <u>linearly independent</u>.

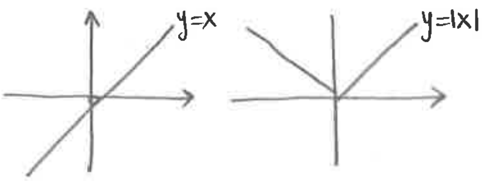
Ex: $f_1(x) = x^2$; $f_2(x) = 3x^2$

$-3f_1(x) + f_2(x) = -3x^2 + 3x^2 = 0, \ \forall x \in \mathbb{R}$

$\Rightarrow$ linearly dependent on $\mathbb{R}$

Two functions $f_1(x), f_2(x)$ are lin. dep. if and only if they are constant multiples of one another.

Ex: $f_1(x) = x$; $f_2(x) = |x|$



$\rightarrow$ lin. indp. on $\mathbb{R}$ (cannot be expressed as constant multiples of each other on $\mathbb{R}$)

$\rightarrow$ lin. dep. on $(0, \infty)$ $\quad x = 1 \cdot |x|$ on $(0, \infty)$

$\rightarrow$ lin. dep. on $(-\infty, 0)$ $\quad x = -1 \cdot |x|$ on $(-\infty, 0)$.

Ex: $f_1(x) = \sqrt{x} + 3$

$f_2(x) = \sqrt{x} + 3x$

$f_3(x) = x - 1$

$f_4(x) = x^2$

$(f_2 - f_1)(x) = 3x - 3$

$= 3(x-1) = 3f_3(x)$

$\Rightarrow f_1(x) - f_2(x) + 3f_3(x) + 0 \cdot f_4(x) = 0 \ \Rightarrow$ lin. dep. on $(0, \infty)$

the constants can't all be 0 but it's allowed for some to be 0.

Wronskian

Suppose $f_1, f_2, \dots, f_n$ have at least $(n-1)$ derivatives. Define their Wronskian as:

$$W(f_1, \dots, f_n)(x) = \begin{vmatrix} f_1(x) & f_2(x) & \dots & f_n(x) \\ f_1'(x) & f_2'(x) & \dots & f_n'(x) \\ \vdots & & & \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \dots & f_n^{(n-1)}(x) \end{vmatrix}$$

- If $f_1, f_2, \dots, f_n$ are <u>linearly dependent</u> on I, then $W(f_1, \dots, f_n)(x) = 0, \ \forall x \in I$.

$\Rightarrow$ If $W(f_1, \dots, f_n)(x) \neq 0$ for at least one value of x on I, then $f_1, \dots, f_n$ are <u>linearly independent</u>.

Proof ($n=2$): Suppose $W(f_1, f_2)(x) \neq 0$ for at least one $x \in I$, and also that f_1, f_2 are lin. dep. $\Rightarrow \exists$ constants (not both 0) c_1, c_2 s.t. $c_1 f_1 + c_2 f_2 = 0$ on I.

Assume $c_1 \neq 0$.

$c_1 f_1 + c_2 f_2 = 0 \ | \cdot f_2' \qquad c_1 f_1 f_2' + c_2 f_2 f_2' = 0$

$c_1 f_1' + c_2 f_2' = 0 \ | \cdot f_2 \qquad c_1 f_1' f_2 + c_2 f_2 f_2' = 0$

$\ominus \quad c_1 (f_1 f_2' - f_1' f_2) = 0$

$\Rightarrow \underbrace{f_1 f_2' - f_1' f_2}_{W(f_1, f_2)} = 0$

$W(f_1, f_2) = 0$

$\Rightarrow$ contradiction.