Exam 1 Review Problems

1 $y\frac{dy}{dx} = (x+xy^2)e^{x^2}$

$y\frac{dy}{dx} = x(1+y^2)e^{x^2}$ Separable

$$\int \frac{y}{1+y^2}dy = \int xe^{x^2}dx$$

$$\frac{1}{2}\ln(1+y^2) = \frac{1}{2}e^{x^2}+c$$

$$\ln(1+y^2) = e^{x^2}+c$$

$$\boxed{1+y^2 = ce^{e^{x^2}}}$$

$$\left[\begin{array}{l} y\,dy = (x+xy^2)e^{x^2}dx \\ (x+xy^2)e^{x^2}dx - y\,dy = 0 \\ M_y = 2xe^{x^2}y \;;\; N_x = 0 \Rightarrow \text{not exact} \end{array}\right]$$

2 $\left(x^2+\frac{2y}{x}\right)dx = (3-\ln x^2)dy$

$$\left(x^2+\frac{2y}{x}\right)dx + (\ln x^2-3)dy = 0$$

$M_y = \frac{2}{x}$; $N_x = \frac{1}{x^2}\cdot 2x = \frac{2}{x}$ => Exact

$$\frac{\partial f}{\partial x} = x^2+\frac{2y}{x} \Rightarrow f(x,y) = \frac{1}{3}x^3 + 2y\ln|x| + g(y)$$

$$= \frac{1}{3}x^3 + y\ln(x^2) + g(y)$$

$$\Rightarrow \left.\begin{array}{l}\frac{\partial f}{\partial y} = \ln(x^2)+g'(y) \\ = \ln(x^2)-3\end{array}\right\} \Rightarrow g'(y) = -3 \Rightarrow g(y) = -3y$$

$$\Rightarrow \boxed{\frac{1}{3}x^3 + y\ln(x^2) - 3y = c}$$

$$y(\ln(x^2)-3) = c - \frac{1}{3}x^3 \Rightarrow \boxed{y = \frac{c-1/3x^3}{\ln(x^2)-3}}$$

(2) $x^2 + \frac{2y}{x} + (\ln x^2 - 3)\frac{dy}{dx} = 0$

$$\frac{dy}{dx}(\ln x^2 - 3) + \frac{2}{x}y = -x^2$$ Linear

$$\frac{dy}{dx} + \frac{2}{x}\frac{1}{\ln x^2 - 3}y = -\frac{x^2}{\ln x^2 - 3}$$

$$p(x) = \frac{1}{x}\frac{2}{\ln(x^2) - 3} \Rightarrow \int p(x)dx = \int \frac{2}{x}\frac{1}{2\ln x - 3}dx = \ln|2\ln x - 3|$$

$$\Rightarrow \mu(x) = 2\ln x - 3$$

$$\Rightarrow \frac{d}{dx}\Big(y(2\ln x - 3)\Big) = -x^2 \Rightarrow y(2\ln x - 3) = -\frac{1}{3}x^3 + c$$

$$\Rightarrow \boxed{y = \frac{c - \frac{1}{3}x^3}{\ln(x^2) - 3}}$$

(3) $\frac{dy}{dx} = \frac{x}{y} + \frac{y}{x} + 1$ Homogeneous

$\boxed{u = \frac{y}{x}}$ $\Rightarrow \cancel{u} + x\frac{du}{dx} = \frac{1}{u} + \cancel{u} + 1 \Rightarrow x\frac{du}{dx} = \frac{u+1}{u}$

$dy = u\,dx + x\,du$

$\frac{dy}{dx} = u + x\frac{du}{dx}$

$$\Rightarrow \frac{u}{u+1}du = \frac{1}{x}dx$$

$$\Rightarrow \ln|x| = \int \frac{u+1-1}{u+1}du = \int 1 - \frac{1}{u+1}du$$

$$= u - \ln|u+1| + C$$

$$\Rightarrow \ln|x| + \ln|u+1| - \ln|c| = u$$

$$\Rightarrow \ln\left|\frac{x}{c}\left(\frac{y}{x} + 1\right)\right| = \frac{y}{x}$$

$$\Rightarrow \frac{1}{c}(y + x) = e^{y/x} \Rightarrow \boxed{y + x = ce^{y/x}}$$

Verify: $y' + 1 = ce^{y/x} \cdot \frac{y'x - y}{x^2}$

$x^2(y'+1) = (y+x)(y'x - y)$

$\cancel{x^2y'} + x^2 = xy \cdot y' + \cancel{x^2y'} - y^2 - xy$

$xy \cdot y' = x^2 + y^2 + xy \Rightarrow y' = \frac{x}{y} + \frac{y}{x} + 1$ ✓

(4) $2xyy' + y^2 = 2x^2$

$$2xy\,dy + (y^2 - 2x^2)\,dx = 0$$

$N_x = 2y \qquad M_y = 2y \quad \Rightarrow$ exact

Potential: $\frac{\partial f}{\partial x} = y^2 - 2x^2 \Rightarrow f(x,y) = y^2x - \frac{2}{3}x^3 + g(y)$

$$\Rightarrow \frac{\partial f}{\partial y} = 2xy + g'(y)$$
$$= 2xy$$

$$\Rightarrow \boxed{y^2x - \frac{2}{3}x^3 = C}$$

$2xy\frac{dy}{dx} - 2x^2 = -y^2 \quad /:2xy$

$\frac{dy}{dx} - \frac{x}{y} = -\frac{y}{2x} \Rightarrow \boxed{\frac{dy}{dx} + \frac{1}{2x}y = xy^{-1}}$ Bernoulli w/ $\alpha = -1$

$1 - \alpha = 2$

$\Rightarrow$ (linear) $\frac{du}{dx} + \frac{1}{x}u = 2x \leftarrow \boxed{u = y^2}$

$\mu(x) = x \Rightarrow \frac{d}{dx}(ux) = 2x^2 \Rightarrow ux = \frac{2}{3}x^3 + C$

$$\Rightarrow \boxed{y^2x = \frac{2}{3}x^3 + C}$$

$2xy\,dy + (y^2 - 2x^2)\,dx = 0$ homogeneous (degree 2)

$2\,y/x\,\frac{dy}{dx} + (y/x)^2 - 2 = 0$

$\boxed{u = y/x} \rightarrow 2u\left(u + x\frac{du}{dx}\right) + u^2 - 2 = 0$

$\frac{dy}{dx} = u + x\frac{du}{dx}$

$2u^2 + 2ux\frac{du}{dx} + u^2 - 2 = 0$

$2ux\,du = (-3u^2 + 2)\,dx$

$\frac{2u}{-3u^2+2}du = \frac{1}{x}dx \Rightarrow \ln|x| = -\frac{1}{3}\ln|2 - 3u^2| + C$

$\ln\left|x(2 - 3\,y^2/x^2)^{1/3}\right| = C$

$x\left(\frac{2x^2 - 3y^2}{x^2}\right)^{1/3} = C$

$x^3\,\frac{2x^2 - 3y^2}{x^2} = C$

$x(2x^2 - 3y^2) = C$

$2x^3 + C = 3xy^2$

$\boxed{y^2x = \frac{2}{3}x^3 + C}$ ✓

-6

5. $\frac{y}{x^2}\frac{dy}{dx} + e^{2x^3+y^2} = 0$

$$\frac{y}{x^2}\frac{dy}{dx} = -e^{2x^3}e^{y^2}$$

$$y\,dy = -x^2e^{2x^3}e^{y^2}dx \qquad \boxed{ye^{-y^2}dy = -x^2e^{2x^3}dx}$$

Separable

$$\int ye^{-y^2}dy = -\frac{1}{2}e^{-y^2}$$

$$\int x^2e^{2x^3}dx = \frac{1}{6}e^{2x^3}$$

$$\Rightarrow -\frac{1}{2}e^{-y^2} = -\frac{1}{6}e^{2x^3}+c$$

$$\boxed{3e^{-y^2} = e^{2x^3}+c}$$

Check: $-6e^{-y^2}\cdot y\cdot y' = 6x^2e^{2x^3}$

$$e^{-y^2}\cdot y\cdot y' + x^2e^{2x^3} = 0$$

$$y\cdot y' + x^2e^{2x^3+y^2} = 0$$

$$\frac{y}{x^2}y' + e^{2x^3+y^2} = 0 \quad ✓$$

6. $xyy' + y^2 = 2x$ $\quad /:xy$

$$y' + \frac{1}{x}y = \frac{2}{y}$$

$$y' + \frac{1}{x}y = 2y^{-1}$$

Bernoulli w/ $\alpha = -1$

$1-\alpha = 2$

$$\boxed{u = y^2} \rightarrow \frac{du}{dx} + \frac{2}{x}u = 4$$

$$\mu(x) = x^2 \Rightarrow \frac{d}{dx}(ux^2) = 4x^2$$

$$ux^2 = \frac{4}{3}x^3 + c$$

$$u = \boxed{y^2 = \frac{4}{3}x + \frac{c}{x^2}}$$

Check: $2yy' = \frac{4}{3} - \frac{2c}{x^3}$

$$xyy' = \frac{2x}{3} - \frac{c}{x^2}$$

$$xyy' + \underbrace{\frac{4}{3}x + \frac{c}{x^2}}_{y^2} = 2x$$

$$xyy' + y^2 = 2x \quad ✓$$

7) $y\,dx + x\,dy = 0$

$$y\,dx = -x\,dy$$

$$-\frac{1}{x}dx = \frac{1}{y}dy \quad \underline{\text{Separable}}$$

$$C - \ln|x| = \ln|y|$$

$$\frac{C}{|x|} = |y| \Rightarrow \boxed{y = \frac{C}{x}}$$

$$M_y = N_x = 1 \Rightarrow \underline{\text{Exact}}$$

Potential: $\frac{\partial f}{\partial x} = y \Rightarrow f(x,y) = xy + g(y)$

$$\Rightarrow \frac{\partial f}{\partial y} = x + \underbrace{g'(y)}_{=0}$$

$$\Rightarrow \boxed{xy = C}$$

8) $\frac{dy}{dx} = \frac{1}{y-x}$

Not linear (in y)

$(y-x)\,dy = dx$ — Not separable

$dx + (x-y)\,dy = 0$ — $M_y = 0$; $N_x = 1$ Not exact

$\underbrace{1}_{\text{hom. of deg. } 0}\,dx + \underbrace{(x-y)}_{\text{hom. of degree } 1}\,dy = 0$ — Not homogeneous

Not Bernoulli

Reciprocal Trick: $\frac{dx}{dy} = y - x \Rightarrow \boxed{\frac{dx}{dy} + x = y}$ linear in x

think of x as a function of y

$p(y) = 1$

$\int p(y)\,dy = y$

$$\mu(y) = e^y \Rightarrow \frac{d}{dy}(xe^y) = ye^y$$

$$xe^y = \int ye^y\,dy$$

$$= ye^y - e^y + C$$

$$\Rightarrow \boxed{x = y - 1 + Ce^{-y}}$$

Does not always work, but is worth trying when you have things like $\frac{dy}{dx} =$ ☺/☆ that you can't solve

Check: $1 = y' - Ce^{-y} \cdot y'$

$$1 = y'(\underbrace{1 - Ce^{-y}}_{y-x}) \Rightarrow y' = \frac{1}{y-x}$$

✓

9) $e^{x+y}\,dy - dx = 0$

$$e^x e^y\,dy = dx$$

$$e^y\,dy = e^{-x}\,dx \qquad \text{Separable}$$

$$e^y = -e^{-x} + c$$

$$\boxed{y = \ln(c - e^{-x})}$$

10) $(x^2-1)\dfrac{dy}{dx} + 2y = (x+1)^2$ Linear

$$\frac{dy}{dx} + \frac{2}{x^2-1}y = \frac{(x+1)^2}{x^2-1}$$

$$\int p(x)dx = \int \frac{2}{x^2-1}dx = \int \frac{2}{(x-1)(x+1)}dx = \int \frac{(x+1)-(x-1)}{(x-1)(x+1)}dx = \int \frac{1}{x-1} - \frac{1}{x+1}\,dx$$

$$= \ln|x-1| - \ln|x+1| = \ln\left|\frac{x-1}{x+1}\right|$$

$$\Rightarrow \mu(x) = \frac{x-1}{x+1} \Rightarrow \frac{d}{dx}\left(y\,\frac{x-1}{x+1}\right) = \frac{(x+1)^2}{x^2-1}\cdot\frac{x-1}{x+1} = \frac{x+1}{x+1} = 1$$

$$y\,\frac{x-1}{x+1} = x + C \Rightarrow \boxed{y = \frac{x+1}{x-1}(x+C)}$$

$$y' = \frac{\cancel{x}-1-\cancel{x}-1}{(x-1)^2}(x+c) + \frac{x+1}{x-1}$$

$$y' = \frac{x^2-1-2(x+c)}{(x-1)^2}$$

$$(x-1)^2 y' = x^2-1-2y\,\frac{x-1}{x+1}$$

$$(x-1)y' = x+1-\frac{2y}{x+1}$$

$$(x^2-1)y' = (x+1)^2 - 2y \quad \checkmark$$

11 $\dfrac{dy}{dx} = \dfrac{y-x}{y+x}$

$$\frac{dy}{dx} = \frac{y/x - 1}{y/x + 1}$$ Homogeneous

$\boxed{u = \frac{y}{x}}$

$\frac{dy}{dx} = u + x\frac{du}{dx}$

$$u + x\frac{du}{dx} = \frac{u-1}{u+1} \Rightarrow x\frac{du}{dx} = \frac{\cancel{u}-1-u^2-\cancel{u}}{u+1}$$

$$\Rightarrow \frac{u+1}{u^2+1}du = -\frac{1}{x}dx$$

$$\Rightarrow -\ln|x| = \int \frac{u+1}{u^2+1}du = \int \frac{u}{u^2+1}du + \int \frac{1}{u^2+1}du$$

$$= \frac{1}{2}\ln|u^2+1| + \arctan(u) + C$$

$$\Rightarrow \ln|u^2+1| + \ln|x^2| + \ln|c| = -2\arctan(u)$$

$$\Rightarrow \ln\left|cx^2\left(\frac{y^2}{x^2}+1\right)\right| = -2\arctan(y/x)$$

$$\boxed{c(x^2+y^2) = e^{-2\arctan(y/x)}}$$

Check:

$$c(2x+2yy') = e^{-2\arctan(y/x)}(-2)\cdot\frac{1}{1+(y/x)^2}\cdot\frac{y'x-y}{x^2}$$

$$\cancel{c}(2x+2yy') = \cancel{c}(x^2+y^2)(-2)\cdot\frac{\cancel{x^2}}{\cancel{x^2+y^2}}\cdot\frac{y'x-y}{\cancel{x^2}}$$

$$x + yy' = -y'x + y$$

$$y'(y+x) = y - x$$

$$y' = \frac{y-x}{y+x} \checkmark$$

12 $x\,dy = y\ln y\,dx\ ;\quad y(2)=e$

Separable : $\dfrac{1}{x}dx = \dfrac{1}{y\ln y}dy$

$\ln|x| + C = \ln|\ln y|$

$C|x| = |\ln y|$

$cx = \ln y$

$\boxed{e^{cx}} = y \xrightarrow[x=2,\,y=e]{} e^{2c} = e \Rightarrow 2c = 1 \Rightarrow \boxed{c=1/2}$

$\Rightarrow \boxed{y = e^{x/2}}\quad (-\infty, \infty)$

13 $\left(1+\ln x+\dfrac{y}{x}\right)dx = (1-\ln x)\,dy$

$\left(1+\ln x+\dfrac{y}{x}\right)dx + (\ln x - 1)\,dy = 0$

! Be sure to put eqn. in the form $Mdx+Ndy=0$ before testing for exactness! (i.e. don't test $Mdx = Ndy$)

$\dfrac{\partial M}{\partial y} = \dfrac{1}{x}\ ;\quad \dfrac{\partial N}{\partial x} = \dfrac{1}{x} \Rightarrow$ Exact

Potential: $\dfrac{\partial f}{\partial x} = 1+\ln x+\dfrac{y}{x} \Rightarrow f(x,y) = x + (x\ln x - x) + y\ln x + g(y) = x\ln x + y\ln x + g(y)$

$\Rightarrow \left.\begin{aligned}\dfrac{\partial f}{\partial y} &= \ln x + g'(y)\\ &= \ln x - 1\end{aligned}\right\} \Rightarrow g(y) = -y$

$\boxed{x\ln x + y\ln x - y = C}$

Check: $\ln x + 1 + y'\ln x + \dfrac{y}{x} - y' = 0$

$(\ln x - 1)dy + \left(1+\ln x+\dfrac{y}{x}\right)dx = 0$ ✓

14 $xy' + y = e^x\ ;\quad y(1) = 2$ Linear

$y' + \dfrac{1}{x}y = \dfrac{1}{x}e^x$

$\mu(x) = x \Rightarrow \dfrac{d}{dx}(xy) = e^x \Rightarrow xy = e^x + C \Rightarrow y = \dfrac{1}{x}e^x + \dfrac{C}{x}$

$y(1) = 2 \Rightarrow 2 = e + C$

$\Rightarrow \boxed{y = \dfrac{1}{x}(e^x + 2 - e)}\quad (0, \infty)$

need to take the interval containing 1 b/c of init. cond.

15 $dy - \sin x\,(y+2)\,dx = 0 \quad ; \; y(\pi/2) = 1$

Exact? $M_y = -\sin x \; ; \; N_x = 1$ No.

Homogeneous? No

Separable?

$$dy = \sin x\,(y+2)\,dx$$

$$\frac{1}{y+2}\,dy = \sin x\,dx \qquad \text{Yes!}$$

$$\ln|y+2| = -\cos x + c$$

$$|y+2| = c\,e^{-\cos x} \quad \Rightarrow \quad y+2 = c\,e^{-\cos x}$$

$$x = \pi/2 \; ; \; y = 1 \Rightarrow 3 = c \Rightarrow \boxed{y = 3e^{-\cos x} - 2}$$

$(-\infty, \infty)$

Linear?

$$\frac{dy}{dx} - \sin x\,(y+2) = 0$$

$$\frac{dy}{dx} - (\sin x)\,y = 2\sin x \qquad \text{Yes}$$

$$\mu(x) = e^{\cos x} \Rightarrow \frac{d}{dx}\left(y e^{\cos x}\right) = 2\sin x\, e^{\cos x}$$

$$\Rightarrow y e^{\cos x} = -2 e^{\cos x} + c$$

$$\Rightarrow y = -2 + c e^{-\cos x} \Rightarrow \boxed{y = 3e^{-\cos x} - 2}$$

16 $\dfrac{dy}{dx} = \dfrac{xy + 3x - y - 3}{xy - 2x + 4y - 8}$ (Note: not homogeneous b/c of the (-3) & (-8))

$$\frac{dy}{dx} = \frac{(x-1)(y+3)}{(x+4)(y-2)} \qquad \text{Separable: } \frac{y-2}{y+3}\,dy = \frac{x-1}{x+4}\,dx$$

$$\int \frac{y-2}{y+3}\,dy = \int \frac{(y+3)-5}{y+3}\,dy = \int\left(1 - \frac{5}{y+3}\right)dy = y - 5\ln|y+3|$$

$$\int \frac{x-1}{x+4}\,dx = \int \frac{(x+4)-5}{(x+4)}\,dx = \int\left(1 - \frac{5}{x+4}\right)dx = x - 5\ln|x+4|$$

$$y - 5\ln|y+3| = x - 5\ln|x+4| + c$$

$$y - x + c = 5\ln\left|\frac{y+3}{x+4}\right|$$

$$c\,e^{y-x} = \left|\frac{y+3}{x+4}\right|^5$$

$$\boxed{\left(\frac{y+3}{x+4}\right)^5 = c\,e^{y-x}}$$

(17) $y\left(\ln\left(\frac{y}{x}\right)+1\right)dx - x\,dy = 0$

Exact? $M_y = \ln\left(\frac{y}{x}\right)+1+ y\cdot\frac{\cancel{x}}{\cancel{y}}\cdot\frac{1}{\cancel{x}}$; $N_x = -1$ No

Linear? No – you'll have a $\ln(y)$ term

Separable? No – you won't be able to separate the term $(\ln y - \ln x + 1)$

Homogeneous? Yes – degree 1.

How can you see this?

$$\underbrace{y}_{1}\underbrace{\left(\ln\left(\frac{y}{x}\right)+1\right)}_{0}dx - \underbrace{x}_{1}dy = 0$$

This term is homogeneous of degree 0

Solve: (divide by $x\,dx$) $\frac{y}{x}\left(\ln\left(\frac{y}{x}\right)+1\right) - \frac{dy}{dx} = 0$

$\boxed{u = \frac{y}{x}}$ $\rightsquigarrow$ $u(\ln u + 1) - (u + xu') = 0$

$\boxed{y' = u + xu'}$

$$u\ln u + \cancel{u} - \cancel{u} = x\frac{du}{dx}$$

$$\frac{1}{x}dx = \frac{1}{u\ln u}du$$

$$\ln|x| = \ln|\ln u| + c$$

$$\ln|x| = \ln\left|\ln\frac{y}{x}\right| + c$$

$$x = c\ln\frac{y}{x} \Rightarrow cx = \ln\frac{y}{x} \Rightarrow \frac{y}{x} = e^{cx}$$

$$\Rightarrow \boxed{y = xe^{cx}}$$

Check: $y' = e^{cx} + cxe^{cx}$

$$y' = \underbrace{e^{cx}}_{y/x}\underbrace{(1+cx)}_{1+\ln(y/x)}$$

✓

18. $\frac{dy}{dx} = \frac{2\ln x}{xy}$ Separable

$$\int y\,dy = \int \frac{2\ln x}{x}\,dx$$

$$y^2/2 = (\ln x)^2 + C$$

$$\boxed{y^2 = 2(\ln x)^2 + C}$$

Check: $2yy' = 4(\ln x)\cdot\frac{1}{x}$

$$y' = \frac{2\ln x}{xy} \checkmark$$

19. $\frac{dy}{dx} + \frac{1}{x}y = \frac{25x^2\ln x}{2y}$ Bernoulli w/ $\alpha = -1$

$1 - \alpha = 2$

$$\frac{dy}{dx} + \frac{1}{x}y = \frac{25x^2\ln x}{2}\cdot y^{-1}$$

$\boxed{u = y^2}$ $\quad \frac{du}{dx} + \frac{2}{x}u = 25x^2\ln x$

$$\mu(x) = x^2 \Rightarrow \frac{d}{dx}(ux^2) = 25x^4\ln x$$

$$\Rightarrow ux^2 = \int 25x^4\ln x\,dx = \int 5(x^5)'\ln x\,dx$$

$$= 5\left(x^5\ln x - \int x^5\cdot\frac{1}{x}dx\right)$$

$$= 5x^5\ln x - x^5 + C$$

$$\Rightarrow u = \boxed{y^2 = 5x^3\ln x - x^3 + \frac{C}{x^2}}$$

Check: $2yy' = 15x^2\ln x + 5x^2 - 3x^2 - \frac{2C}{x^3}$

$$\frac{1}{x}2y^2 = 10x^2\ln x - 2x^2 + \frac{2C}{x^3}$$

$$2y\left(y' + \frac{1}{x}y\right) = 25x^2\ln x \checkmark$$

(20) $\frac{dy}{dx} + \frac{2e^{2x}}{1+e^{2x}} y = \frac{1}{e^{2x}-1}$ Linear

$$\int p(x)dx = \int \frac{2e^{2x}}{1+e^{2x}}dx = \ln(1+e^{2x}) \Rightarrow \mu(x) = 1+e^{2x}$$

$$\Rightarrow \frac{d}{dx}\left[y(1+e^{2x})\right] = \frac{1+e^{2x}}{e^{2x}-1}$$

$$y(1+e^{2x}) = \int \frac{e^{2x}+1}{e^{2x}-1}\bigg|\begin{matrix}\cdot e^{-x}\\ \cdot e^{-x}\end{matrix}\, dx = \int \frac{e^{x}+e^{-x}}{e^{x}-e^{-x}}dx = \ln|e^{x}-e^{-x}|+c$$

$$y = \frac{1}{(1+e^{2x})}\left(\ln|\underbrace{e^{x}-e^{-x}}_{\frac{e^{2x}-1}{e^{x}}}|+c\right) = \boxed{\frac{1}{(1+e^{2x})}\left(\ln|e^{2x}-1|-x+c\right)}$$

$$y' = -\frac{1}{(1+e^{2x})^2}\,2e^{2x}\left(\ln|e^{2x}-1|-x+c\right) + \frac{1}{(1+e^{2x})}\left(\frac{2e^{2x}}{e^{2x}-1}-1\right)$$

$$y' = -\frac{2e^{2x}}{(1+e^{2x})}y + \frac{1}{\cancel{1+e^{2x}}}\,\frac{\cancel{e^{2x}+1}}{e^{2x}-1}$$ ✓

(21) $\frac{dy}{dx} = \frac{\sin y + y\cos x + 1}{1 - x\cos y - \sin x}$ (Not separable or homogeneous) (Not linear b/c of terms like $\sin y$)

Exact? $(\sin y + y\cos x + 1)dx + (\sin x + x\cos y - 1)dy = 0$

$M_y = \cos y + \cos x$; $N_x = \cos x + \cos y$ ✓ Yes.

Potential: $\frac{\partial f}{\partial x} = \sin y + y\cos x + 1 \Rightarrow f(x,y) = x\sin y + y\sin x + x + g(y)$

$$\Rightarrow \left.\begin{aligned}\frac{\partial f}{\partial y} &= x\cos y + \sin x + g'(y)\\ &= x\cos y + \sin x - 1\end{aligned}\right\} \Rightarrow g(y) = -y$$

$$\Rightarrow \boxed{x\sin y + y\sin x + x - y = c}$$

$\sin y + x\cos y \cdot y' + y'\sin x + y\cos x + 1 - y' = 0$

$(x\cos y + \sin x - 1)y' + (\sin y + y\cos x + 1) = 0$ ✓

22 $\left(\ln(xy)+1\right)dx+\left(\frac{x}{y}+2y\right)dy=0$

Exact? $M_y=\frac{x}{xy}=\frac{1}{y}$; $N_x=\frac{1}{y}$ Yes

Potential: $\frac{\partial f}{\partial x}=\ln(xy)+1=\ln x+\ln y+1$

$\Rightarrow f(x,y)=x\ln x+x\ln y+g(y)$

$$\left.\begin{aligned}\Rightarrow \frac{\partial f}{\partial y}&=\frac{x}{y}+g'(y)\\ &=\frac{x}{y}+2y\end{aligned}\right\}\Rightarrow g(y)=y^2 \Rightarrow \boxed{x\ln(xy)+y^2=c}$$

$\ln x+1+\ln y+\frac{x}{y}y'+2yy'=0$

$\ln(xy)+1+\left(\frac{x}{y}+2y\right)y'=0$ ✓

23 $\frac{dy}{dx}-x^2y=\sqrt{y}\,x^2$ Bernoulli w/ $\alpha=1/2$

$1-\alpha=1/2$

$\boxed{u=\sqrt{y}}$ $\frac{du}{dx}-\frac{1}{2}x^2u=\frac{1}{2}x^2$

$\mu(x)=e^{-1/6x^3} \Rightarrow \frac{d}{dx}\left(u\,e^{-1/6x^3}\right)=\frac{1}{2}e^{-1/6x^3}x^2$

$\Rightarrow \sqrt{y}\,e^{-1/6x^3}=-e^{-1/6x^3}+c \Rightarrow \sqrt{y}=-1+ce^{1/6x^3}$

$\Rightarrow \boxed{y=\left(-1+ce^{1/6x^3}\right)^2}$

$y'=2\left(-1+ce^{1/6x^3}\right)c\cdot\frac{x^2}{2}e^{1/6x^3}=\underbrace{ce^{1/6x^3}}_{\sqrt{y}+1}\underbrace{\left(-1+ce^{1/6x^3}\right)}_{\sqrt{y}}x^2=x^2\sqrt{y}\left(\sqrt{y}+1\right)=x^2y+x^2\sqrt{y}$

$y'-x^2y=x^2\sqrt{y}$ ✓

24 $2x(\ln x)y' - y = -9x^3y^3\ln x$ Bernoulli w/ $\alpha = 3$

$1-\alpha = -2$

$$y' - \frac{1}{2x(\ln x)}y = -\frac{9}{2}x^2y^3$$

$$\boxed{u = y^{-2}} \Rightarrow \frac{du}{dx} + \frac{1}{x\ln x}u = 9x^2$$

$$\mu(x) = e^{\ln(\ln x)} = \ln x \Rightarrow \frac{d}{dx}(u\ln x) = 9x^2\ln x$$

$$\Rightarrow y^{-2}\ln x = \int 9x^2\ln x\,dx = \int 3(x^3)'\ln x\,dx$$

$$= 3x^3\ln x - x^3 + C$$

$$\Rightarrow \boxed{y^{-2} = 3x^3 - \frac{x^3 + c}{\ln x}}$$

$$-2y^{-3}y' = 9x^2 - \frac{3x^2\ln x - x^2 - c/x}{(\ln x)^2}$$

$$2x(\ln x)y' = y^3\Big(\underbrace{\frac{3x^3\ln x - x^3 - c}{\ln x}}_{\frac{1}{y^2}} - 9x^3\ln x\Big)$$

$$2x(\ln x)y' - y = -9x^3y^3\ln x \checkmark$$

25 $e^{2x+y}dy - e^{x-y}dx = 0$ Separable

$$e^{2x}e^{y}dy = e^{x}e^{-y}dx$$

$$e^{2y}dy = e^{-x}dx$$

$$\frac{1}{2}e^{2y} = -e^{-x} + c$$

$$\boxed{e^{2y} = c - 2e^{-x}}$$

$$2e^{2y}y' = +2e^{-x}$$

$$e^{2y}dy - e^{-x}dx = 0$$

$$e^{2y+2x}dy - e^{x}dx = 0$$

$$e^{2x+y}dy - e^{x-y}dx = 0 \checkmark$$

26 $y' + y\sin x = \sin x$ Linear

$$\mu(x) = e^{-\cos x}$$

$$\frac{d}{dx}(ye^{-\cos x}) = \sin x\, e^{-\cos x}$$

$$ye^{-\cos x} = e^{-\cos x} + c$$

$$\boxed{y = 1 + ce^{\cos x}}$$

$$y' = c(-\sin x)e^{\cos x} = -\sin x(y-1)$$

$$y' + \sin x \cdot y = \sin x \checkmark$$

27 $y' + y(\tan x + y\sin x) = 0$

$$y' + y\tan x = -\sin x\, y^2 \qquad \text{Bernoulli w/ } \alpha = 2, \quad 1-\alpha = -1$$

$$\boxed{u = \frac{1}{y}} \Rightarrow \frac{du}{dx} - (\tan x)u = \sin x$$

$$\mu(x) = e^{-\int \tan x\,dx} = e^{\ln|\cos x|} = |\cos x| \ ; \text{ take } \mu(x) = \cos x$$

$$\underbrace{(\cos x)u' - (\sin x)u}_{} = \sin x\cos x$$

$$\frac{d}{dx}\left[(\cos x)u\right] = \sin x\cos x \Rightarrow (\cos x)u = \frac{1}{2}\sin^2 x + c$$

$$\Rightarrow u = \frac{1}{y} = \frac{1}{2}\frac{\sin^2 x}{\cos x} + \frac{c}{\cos x} = \frac{\sin^2 x + c}{2\cos x}$$

$$\boxed{y = \frac{2\cos x}{\sin^2 x + c}}$$

$$y' = \frac{-2\sin x(\sin^2 x + c) - 4\sin x\cos^2 x}{(\sin^2 x + c)^2}$$

$$y\tan x = \frac{2\sin x}{(\sin^2 x + c)}$$

$$\oplus\ y' + y\tan x = \frac{-4\sin x\cos^2 x}{(\sin^2 x + c)^2} = -\sin x \cdot y^2 \checkmark$$

28 $\dfrac{dy}{dx} = \dfrac{x^2}{x^2 - y^2} + \dfrac{y}{x}$ Homogeneous

$$\frac{dy}{dx} = \frac{1}{1 - (y/x)^2} + \frac{y}{x}$$

$$\boxed{u = \frac{y}{x}} \quad \boxed{y' = u + xu'} \Rightarrow \cancel{u} + xu' = \frac{1}{1-u^2} + \cancel{u}$$

$$x\frac{du}{dx} = \frac{1}{1-u^2}$$

$$(1-u^2)\,du = \frac{1}{x}\,dx$$

$$\ln|x| = u - \frac{1}{3}u^3 + c$$

$$|x| = c\,e^{u - \frac{1}{3}u^3}$$

$$\boxed{x = c\,e^{y/x - \frac{1}{3}(y/x)^3}}$$

$$1 = \underbrace{c\,e^{y/x - \frac{1}{3}(y/x)^3}}_{x}\left(1 - \frac{y^2}{x^2}\right)\frac{y'x - y}{x^2}$$

$$1 = \frac{y'x - y}{x}\left(\frac{x^2 - y^2}{x^2}\right)$$

$$y' - \frac{y}{x} = \frac{x^2}{x^2 - y^2} \checkmark$$

29 $(3x^2+2xy^2)\,dx + (2x^2y)\,dy = 0$

Exact? $M_y = 4xy$; $N_x = 4xy$ Yes

Potential: $\frac{\partial f}{\partial x} = 3x^2+2xy^2 \Rightarrow f(x,y) = x^3+x^2y^2+g(y)$

$$\Rightarrow \left.\begin{aligned}\frac{\partial f}{\partial y} &= 2x^2y + g'(y)\\ &= 2x^2y\end{aligned}\right\} \Rightarrow g(y)=0$$

$$\Rightarrow \boxed{x^3+x^2y^2=c}$$

30 $xy' - 2y = 2x^2\ln x$ Linear

$$y' - \frac{2}{x}y = 2x\ln x$$

$$\mu(x) = \frac{1}{x^2} \Rightarrow \frac{d}{dx}\left(y\frac{1}{x^2}\right) = \frac{2\ln x}{x} \Rightarrow y\frac{1}{x^2} = (\ln x)^2 + c$$

$$\boxed{y = x^2(\ln x)^2 + cx^2}$$

31 $(y^2+3xy+x^2)\,dx - x^2\,dy = 0$

Exact? $M_y = 2y+3x$; $N_x = -2x$ No.

Homogeneous? Yes (degree 2)

$$\left(\frac{y^2}{x^2} + \frac{3y}{x} + 1\right) - \frac{dy}{dx} = 0$$

$$\boxed{\begin{aligned} u &= \frac{y}{x}\\ y' &= u + xu'\end{aligned}}$$

$$(u^2+3u+1) = u + x\frac{du}{dx}$$

$$u^2+2u+1 = x\frac{du}{dx}$$

$$\frac{1}{x}\,dx = \frac{1}{(u+1)^2}\,du$$

$$\ln|x| = -\frac{1}{u+1} + c = c - \frac{x}{y+x}$$

$$\boxed{x = c\,e^{-\frac{x}{y+x}}}$$

$$1 = ce^{-\frac{x}{y+x}}\left(-\frac{y+x-x(y'+1)}{(y+x)^2}\right)$$

$$1 = x\,\frac{xy'\cancel{+x}-y\cancel{-x}}{(y+x)^2}$$

$$y^2+2xy+x^2 = x^2y' - xy$$

$$(y^2+3xy+x^2) - x^2y' = 0 \checkmark$$

32 $y' - x^{-1}y = x^{-1}\sqrt{x^2-y^2}$ $(x>0)$

$y' - \frac{y}{x} = \frac{1}{x}\sqrt{x^2-y^2}$ Homogeneous (degree 0)

$y' - \frac{y}{x} = \sqrt{1-(y/x)^2}$

$\boxed{u = y/x}$ $\cancel{u} + xu' - \cancel{u} = \sqrt{1-u^2}$

$y' = u + xu'$ $x\frac{du}{dx} = \sqrt{1-u^2}$

$\frac{1}{\sqrt{1-u^2}}du = \frac{1}{x}dx \Rightarrow \ln|x| = \arcsin(u) + C$

$\Rightarrow \ln x = \arcsin(y/x) + C$

$1 = \underbrace{ce^{\arcsin(y/x)}}_{x}\left(\frac{x}{\sqrt{x^2-y^2}}\ \frac{y'x-y}{x^2}\right)$

$1 = \frac{y'x-y}{\sqrt{x^2-y^2}}$ ✓

$\boxed{x = ce^{\arcsin(y/x)}}$

33 $(1+x)y' = y(2+x)$ Separable

$(1+x)\frac{dy}{dx} = y(2+x)$

$\frac{1}{y}dy = \frac{x+2}{x+1}dx$

$\ln|y| = x + \ln|x+1| + C$

$\ln\left|\frac{y}{x+1}\right| = x + C$

$\frac{y}{x+1} = ce^x$

$\boxed{y = c(x+1)e^x}$

$y' = c(x+1)e^x + ce^x$
$= c(x+2)e^x = \frac{y}{x+1}(x+2)$

✓

34 $(y-e^x)dx + dy = 0$

Exact? $M_y = 1$; $N_x = 0$ No.

$(y-e^x) + \frac{dy}{dx} = 0$ Linear

$\frac{dy}{dx} + y = e^x$; $\mu(x) = e^x$

$\frac{d}{dx}(ye^x) = e^{2x}$; $ye^x = \frac{1}{2}e^{2x} + c$

$\boxed{y = \frac{1}{2}e^x + ce^{-x}}$

$y' = \frac{1}{2}e^x - ce^{-x} = e^x - y$ ✓