First Order Linear Systems

General form (Canonical / normal form - the coefficients of x_i' are 1):

$$\begin{cases} x_1' = p_{11}(t)x_1 + p_{12}(t)x_2 + \dots + p_{1n}(t)x_n + g_1(t) \\ x_2' = p_{21}(t)x_1 + p_{22}(t)x_2 + \dots + p_{2n}(t)x_n + g_2(t) \\ \vdots \qquad\qquad\qquad\qquad \vdots \qquad \vdots \\ x_n' = p_{n1}(t)x_1 + p_{n2}(t)x_2 + \dots + p_{nn}(t)x_n + g_n(t) \end{cases} \qquad t \in I$$

In matrix notation:

$$\boxed{\vec{x}' = P(t)\vec{x} + \vec{g}(t)}$$

where:

$$\vec{x} = \vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

vector of unknown functions $x_1, \dots, x_n$ of the variable t

;

$$\vec{g}(t) = \begin{bmatrix} g_1(t) \\ g_2(t) \\ \vdots \\ g_n(t) \end{bmatrix}$$

the input or forcing function; determines homogeneity of the system:

- if $\vec{g}(t) = \vec{0}$ for all $t \in I$, we say the system is homogeneous
- otherwise, the system is said to be non-homogeneous.

$$P(t) = \begin{bmatrix} p_{11}(t) & \dots & p_{1n}(t) \\ \vdots & & \vdots \\ p_{n1}(t) & \dots & p_{nn}(t) \end{bmatrix}$$

the matrix of coefficients

- A solution is a vector $\vec{x}(t)$ with n component functions, differentiable at all $t \in I$, that satisfies the system.
- If we are also given an initial condition: $\boxed{\vec{x}(t_0) = \vec{x}_0}$ at some $t_0 \in I$, where $\vec{x}_0$ is a constant vector, we have an initial value problem.

Transforming an ODE into a System :

Any n^{th} order linear ODE can be written as a first order linear system :

$$\boxed{\frac{d^n y}{dt^n} + p_1(t)\frac{d^{n-1}y}{dt^{n-1}} + \dots + p_{n-1}(t)\frac{dy}{dt} + p_n(t)y = g(t)}$$

Introduce n variables :

$$\boxed{x_1 = y \ ; \ x_2 = y' \ ; \ x_3 = y'' \ ; \dots ; \ x_n = y^{(n-1)}}$$

$$\Rightarrow \begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ x_3' = x_4 \\ \vdots \\ x_{n-1}' = x_n \\ x_n' = -p_1(t)x_n - \dots - p_{n-1}(t)x_2 - p_n(t)x_1 + g(t) \end{cases}$$

$$\Rightarrow \boxed{\vec{x}' = P(t)\vec{x} + \vec{g}(t)} \quad \text{where} \quad P(t) = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & 0 & \dots & 1 \\ -p_n(t) & -p_{n-1}(t) & \dots & & & -p_1(t) \end{bmatrix} ; \ \vec{g}(t) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ g(t) \end{bmatrix}$$

Example : $\boxed{2y''' - 6y'' + 4y' + y = \sin t}$

$$y''' = 3y'' - 2y' - \frac{1}{2}y + \frac{1}{2}\sin t$$

$$x_1 = y \ ; \ x_2 = y' \ ; \ x_3 = y''$$

$$\begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ x_3' = 3x_3 - 2x_2 - \frac{1}{2}x_1 + \sin t \end{cases}$$

$$\vec{x}' = P(t)\vec{x} + \vec{g}(t) \ , \text{ where } P(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1/2 & -2 & 3 \end{bmatrix} ; \ \vec{g}(t) = \begin{bmatrix} 0 \\ 0 \\ \sin t \end{bmatrix}$$

Existence & Uniqueness :

Consider the initial value problem:

$$\vec{x}' = P(t)\vec{x} + \vec{g}(t) \quad ; \quad \vec{x}(t_0) = \vec{x}_0 \qquad (*)$$

where $t \in I$ for some open interval I. If $P(t)$ and $g(t)$ are continuous on I (i.e. every component function is continuous on I), then $(*)$ has a unique solution $\vec{x}(t)$ on I.

Superposition :

If $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k$ are solutions to the homogeneous linear system $\boxed{\vec{x}' = P(t)\vec{x}}$ on an interval I, then any linear combination $c_1\vec{x}_1 + c_2\vec{x}_2 + \ldots + c_k\vec{x}_k$ is also a solution to the system on I.

Def.: Linear Independence

We say that n vector functions $\vec{x}_1(t), \ldots, \vec{x}_n(t)$ are linearly independent on an interval I provided that

$$\boxed{c_1\vec{x}_1(t) + \ldots + c_n\vec{x}_n(t) = 0, \ \forall t \in I} \iff \boxed{c_1 = c_2 = \ldots = c_n = 0}$$

Otherwise, if there exist constants $c_1, c_2, \ldots, c_n$ not all zero such that

$$c_1\vec{x}_1(t) + \ldots + c_n\vec{x}_n(t) = 0, \ \forall t \in I$$

we say $\vec{x}_1, \ldots, \vec{x}_n$ are linearly dependent on I.

Linear Independence & Wronskian :

Suppose that:

$$\vec{x}_1 = \begin{bmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{bmatrix}, \quad \vec{x}_2 = \begin{bmatrix} x_{12} \\ x_{22} \\ \vdots \\ x_{n2} \end{bmatrix}, \ \ldots, \ \vec{x}_n = \begin{bmatrix} x_{1n} \\ x_{2n} \\ \vdots \\ x_{nn} \end{bmatrix}$$

are n solutions to the homogeneous system $\vec{x}' = P(t)\vec{x}$ on some interval I. Then the set $\{\vec{x}_1, \ldots, \vec{x}_n\}$ of solutions is linearly independent if and only if the Wronskian :

$$W(\vec{x}_1, \ldots, \vec{x}_n) := \begin{vmatrix} x_{11} & x_{12} & \ldots & x_{1n} \\ x_{21} & x_{22} & \ldots & x_{2n} \\ \vdots & & & \vdots \\ x_{n1} & x_{n2} & \ldots & x_{nn} \end{vmatrix}$$

satisfies $W(\vec{x}_1, \ldots, \vec{x}_n)(t) \neq 0$, $\forall t \in I$.

Def.: Fundamental Set of Solutions / Fundamental Matrix

Consider the homogeneous linear system $\vec{x}' = P(t)\vec{x}$, $t \in I$. A fundamental set of solutions to this system on I is any set $\{\vec{x}_1, .., \vec{x}_n\}$ of n linearly independent solutions (where n is the dimension of the system). Given a fundamental set:

$$\vec{x}_1 = \begin{bmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{bmatrix}, \quad \vec{x}_2 = \begin{bmatrix} x_{12} \\ x_{22} \\ \vdots \\ x_{n2} \end{bmatrix}, \quad \dots, \quad \vec{x}_n = \begin{bmatrix} x_{1n} \\ x_{2n} \\ \vdots \\ x_{nn} \end{bmatrix}$$

the matrix:

$$\Phi(t) := \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{bmatrix}$$

is called a fundamental matrix of the system on I.

Def.: General Solution:

Given a fundamental set $\{\vec{x}_1, .., \vec{x}_n\}$ of solutions to the homogeneous system $\vec{x}' = P(t)\vec{x}$, $t \in I$, the general solution to the system is

$$\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2 + \dots + c_n\vec{x}_n = \Phi(t)\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \Phi(t)\vec{c}$$

where $c_1, c_2, .., c_n$ are arbitrary constants.

Remark: If $\Phi(t)$ is the fundamental matrix corresponding to a fundamental set $\{\vec{x}_1, .., \vec{x}_n\}$ of solutions:

$$\boxed{\det(\Phi(t)) = W(\vec{x}_1, .., \vec{x}_n)}$$

By linear independence, $W(\vec{x}_1, .., \vec{x}_n)(t) \neq 0$ for all $t \in I$, so $\det(\Phi(t)) \neq 0$, $\forall t \in I$.
In other words:

> The fundamental matrix $\Phi(t)$ of a homogeneous linear system is invertible for all $t \in I$.

Remark: It can be shown that if $\{\vec{x}_1, .., \vec{x}_n\}$ are any n solutions to $\vec{x}' = P(t)\vec{x}$, then either $W(\vec{x}_1, .., \vec{x}_n)(t) = 0$ for all $t \in I$, or $W(\vec{x}_1, .., \vec{x}_n)(t) \neq 0$ for all $t \in I$. So if we can show $W(\vec{x}_1, .., \vec{x}_n)(t_0) \neq 0$ for some $t_0 \in I$, then $W(\vec{x}_1, .., \vec{x}_n)(t) \neq 0$, for all $t \in I$.

Remark: Suppose we have an IVP: $\vec{x}' = P(t)\vec{x}$; $\vec{x}(t_0) = \vec{x}_0$ on some interval I.

The general solution:

$$\vec{x}(t) = \Phi(t)\vec{c},$$

where $\Phi(t)$ is a fundamental matrix and $\vec{c}$ is a vector of arbitrary constants, must then satisfy:

$$\vec{x}(t_0) = \Phi(t_0)\vec{c} = \vec{x}_0 \Rightarrow \boxed{\vec{c} = \Phi^{-1}(t_0)\vec{x}_0}$$

So the solution to the IVP is given by: $\boxed{\vec{x}(t) = \Phi(t)\Phi^{-1}(t_0)\vec{x}_0}$

Def.: Special Fundamental Matrix:

Let $\vec{x}' = P(t)\vec{x}$ be an n-dimensional linear homogeneous system. Suppose that $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ are solutions to this system satisfying

$$\vec{v}_1(t_0) = \vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} ; \quad \vec{v}_2(t_0) = \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} ; \dots ; \quad \vec{v}_n(t_0) = \vec{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

for some fixed $t_0 \in I$. The special fundamental matrix $\Psi(t)$ is the matrix whose column vectors are $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$.

- $\Psi(t_0) = I_n = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$

This also shows that any homogeneous system $\vec{x}' = P(t)\vec{x}$ has a fundamental set on I.

- $\Psi(t)$ is a fundamental matrix (i.e. $\{\vec{v}_1, \dots, \vec{v}_n\}$ is a fundamental set).

  Since $\det(\Psi(t_0)) = \det(I_n) = 1 \neq 0$, we have that

  $$\det(\Psi(t_0)) = W(\vec{x}_1, \dots, \vec{x}_n)(t_0) \neq 0$$

  So the Wronskian is non-zero for some $t_0 \in I$, therefore it is non-zero everywhere on I.

- $\boxed{\Psi(t) = \Phi(t)\Phi^{-1}(t_0)}$

$$\left.\begin{aligned} \vec{v}_1(t) &= \Phi(t)\Phi^{-1}(t_0)\vec{e}_1 \\ \vec{v}_2(t) &= \Phi(t)\Phi^{-1}(t_0)\vec{e}_2 \\ &\vdots \\ \vec{v}_n(t) &= \Phi(t)\Phi^{-1}(t_0)\vec{e}_n \end{aligned}\right\}$$

$\Rightarrow$ The columns of $\Phi(t)\Phi^{-1}(t_0)$ are the vectors $\vec{v}_1, \dots, \vec{v}_n$, so $\Phi(t)\Phi^{-1}(t_0) = \Psi(t)$.

- In terms of $\Psi(t)$, the solution to the IVP $\boxed{\vec{x}' = P(t)\vec{x} ; \vec{x}(t_0) = \vec{x}_0}$ is $\boxed{\vec{x}(t) = \Psi(t)\vec{x}_0}$

Example: Consider the system $\vec{x}' = \begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix} \vec{x}$ on $\mathbb{R}$.

- Check that $\vec{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-2t}$ and $\vec{x}_2 = \begin{bmatrix} 3 \\ 5 \end{bmatrix} e^{6t}$ are solutions:

$$\vec{x}_1' = -2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-2t} = \begin{bmatrix} -2 \\ 2 \end{bmatrix} e^{-2t} \quad \checkmark$$
$$\begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix} \vec{x}_1 = \begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-2t} = \begin{bmatrix} -2 \\ 2 \end{bmatrix} e^{-2t}$$

$$\vec{x}_2' = 6 \begin{bmatrix} 3 \\ 5 \end{bmatrix} e^{6t} = \begin{bmatrix} 18 \\ 30 \end{bmatrix} e^{6t} \quad \checkmark$$
$$\begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix} \vec{x}_2 = \begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix} e^{6t} = \begin{bmatrix} 18 \\ 30 \end{bmatrix} e^{6t}$$

- Check that $\{\vec{x}_1, \vec{x}_2\}$ is a fundamental set:

$$W(\vec{x}_1, \vec{x}_2)(t) = \begin{vmatrix} 1 & 3 \\ -1 & 5 \end{vmatrix} e^{4t} = 8e^{4t} \neq 0, \ \forall t \in \mathbb{R}$$

- General Solution: $\vec{x} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 3 \\ 5 \end{bmatrix} e^{6t} = \Phi(t)\vec{c}$

- Fundamental Matrix: $\boxed{\Phi(t) = \begin{bmatrix} e^{-2t} & 3e^{6t} \\ -e^{-2t} & 5e^{6t} \end{bmatrix}}$

- Inverse of Fundamental Matrix:

$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ invertible

$\Rightarrow A^{-1} = \frac{1}{\det A} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$

$$\det \Phi(t) = +8e^{4t}$$
$$\Rightarrow \Phi^{-1}(t) = \frac{1}{8e^{4t}} \begin{bmatrix} 5e^{6t} & -3e^{6t} \\ e^{-2t} & e^{-2t} \end{bmatrix}$$
$$\Rightarrow \Phi^{-1}(t) = \begin{bmatrix} 5/8\, e^{2t} & -3/8\, e^{2t} \\ 1/8\, e^{-6t} & 1/8\, e^{-6t} \end{bmatrix}.$$

• Find a special fundamental matrix:

Choose $t_0=0$, for example. Then

$$\Phi^{-1}(0)=\begin{bmatrix} 5/8 & -3/8 \\ 1/8 & 1/8 \end{bmatrix}$$

$$\Rightarrow \Psi(t)=\Phi(t)\,\Phi^{-1}(0)=\begin{bmatrix} e^{-2t} & 3e^{6t} \\ -e^{-2t} & 5e^{6t} \end{bmatrix}\begin{bmatrix} 5/8 & -3/8 \\ 1/8 & 1/8 \end{bmatrix}$$

$$\boxed{\Psi(t)=\begin{bmatrix} 5/8\, e^{-2t}+3/8\, e^{6t} & -3/8\, e^{-2t}+3/8\, e^{6t} \\ -5/8\, e^{-2t}+5/8\, e^{6t} & 3/8\, e^{-2t}+5/8\, e^{6t} \end{bmatrix}}$$

◉ Remark: $\Psi(0)=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}=I_2.$

◉ Remark: $\vec{v}_1(t)=\begin{bmatrix} 5/8\, e^{-2t}+3/8\, e^{6t} \\ -5/8\, e^{-2t}+5/8\, e^{6t} \end{bmatrix}$ is the solution to the system satisfying $\vec{v}_1(0)=\vec{e}_1$.

$\vec{v}_2(t)=\begin{bmatrix} -3/8\, e^{-2t}+3/8\, e^{6t} \\ 3/8\, e^{-2t}+5/8\, e^{6t} \end{bmatrix}$ is the solution to the system satisfying $\vec{v}_2(0)=\vec{e}_2$.

◉ Remark: One may choose a different value of t_0 and obtain a different special fundamental matrix. For example, for $t_0=1$:

$$\Phi^{-1}(1)=\begin{bmatrix} 5/8\, e^{2} & -3/8\, e^{2} \\ 1/8\, e^{-6} & 1/8\, e^{-6} \end{bmatrix}$$

$$\Rightarrow \Psi(t)=\Phi(t)\,\Phi^{-1}(1)=\begin{bmatrix} e^{-2t} & 3e^{6t} \\ -e^{-2t} & 5e^{6t} \end{bmatrix}\begin{bmatrix} 5/8\, e^{2} & -3/8\, e^{2} \\ 1/8\, e^{-6} & 1/8\, e^{-6} \end{bmatrix}=$$

$$=\begin{bmatrix} 5/8\, e^{-2t+2}+3/8\, e^{6t-6} & -3/8\, e^{-2t+2}+3/8\, e^{6t-6} \\ -5/8\, e^{-2t+2}+5/8\, e^{6t-6} & 3/8\, e^{-2t+2}+5/8\, e^{6t-6} \end{bmatrix}$$

Then $\Psi(1)=I_2$, and the first column of $\Psi(t)$ is the solution to the system satisfying $\vec{v}_1(1)=\vec{e}_1$; the second column of $\Psi(t)$ is the solution to the system satisfying $\vec{v}_2(1)=\vec{e}_2$.

Eigenvalues & Eigenvectors:

Def.: Given an $n\times n$ matrix A, a number $\lambda \in \mathbb{C}$ is called an eigenvalue of A if there is a non-zero vector $\vec{v}$ such that $\boxed{A\vec{v} = \lambda \vec{v}}$

Any such vector $\vec{v}$ is called an eigenvector for λ.

We may write

$$A\vec{v} = \lambda\vec{v} \iff A\vec{v} - \lambda\vec{v} = \vec{0} \iff (A - \lambda I)\vec{v} = \vec{0}$$

This equation has non-trivial solutions if and only if $\boxed{\det(A-\lambda I) = 0}$

Characteristic Equation of A

$\Rightarrow$ The eigenvalues of A are the roots of the characteristic equation!

Example: $A = \begin{bmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{bmatrix}$

$$\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 2 & 1 \\ 6 & -1-\lambda & 0 \\ -1 & -2 & -1-\lambda \end{vmatrix} = \begin{vmatrix} 6 & -1-\lambda \\ -1 & -2 \end{vmatrix} - (1+\lambda)\begin{vmatrix} 1-\lambda & 2 \\ 6 & -1-\lambda \end{vmatrix}$$

$$= -12 - 1 - \lambda - (1+\lambda)(-1+\lambda^2-12)$$

$$= -13 - \lambda + 1 - \lambda^2 + 12 + \lambda - \lambda^3 + 12\lambda = -\lambda^3 - \lambda^2 + 12\lambda$$

Char. Eqn.: $\boxed{-\lambda^3 - \lambda^2 + 12\lambda = 0}$

Roots: $\lambda(\lambda^2 + \lambda - 12) = 0 \Rightarrow \lambda(\lambda-3)(\lambda+4) = 0 \Rightarrow \boxed{\lambda \in \{0, 3, -4\}} \leftarrow$ eigenvalues

To find the eigenvectors corresponding to an eigenvalue λ, we must solve the linear system $(A-\lambda I)\vec{v} = \vec{0}$. So, reduce $[A - \lambda I \mid \vec{0}]$:

$\lambda = 0$:

$$[A \mid \vec{0}] = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 6 & -1 & 0 & 0 \\ -1 & -2 & -1 & 0 \end{array}\right] \xrightarrow[R_1+R_3]{-6R_1+R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -13 & -6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{-\frac{1}{13}R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 6/13 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{-2R_2+R_1} \left[\begin{array}{ccc|c} 1 & 0 & 1/13 & 0 \\ 0 & 1 & 6/13 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow$ System becomes $\begin{cases} v_1 + \frac{1}{13}v_3 = 0 \\ v_2 + \frac{6}{13}v_3 = 0 \end{cases}$ $\begin{cases} v_1 = -\frac{1}{13}v_3 \\ v_2 = -\frac{6}{13}v_3 \end{cases}$

$\Rightarrow$ any vector of the form: $\begin{bmatrix} -\frac{1}{13}c \\ -\frac{6}{13}c \\ c \end{bmatrix}$

is an eigenvector for $\lambda = 0$. Choose $c = -13$: $\boxed{\vec{v}_1 = \begin{bmatrix} 1 \\ 6 \\ -13 \end{bmatrix}}$

$\lambda_2 = -4$

$$[A+4I \mid \vec{0}] = \left[\begin{array}{ccc|c} 5 & 2 & 1 & 0 \\ 6 & 3 & 0 & 0 \\ -1 & -2 & 3 & 0 \end{array}\right] \xrightarrow[R_{3-1}]{-R_3} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 6 & 3 & 0 & 0 \\ 5 & 2 & 1 & 0 \end{array}\right] \xrightarrow[-5R_1+R_3]{-6R_1+R_2} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & -9 & +18 & 0 \\ 0 & -8 & 16 & 0 \end{array}\right]$$

$$\xrightarrow[\substack{-\frac{1}{9}R_2 \\ -\frac{1}{8}R_3}]{} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right] \xrightarrow[R_3-R_2]{} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[R_1-2R_2]{} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow$ any vector of the form $\begin{bmatrix} -c \\ 2c \\ c \end{bmatrix}$ is an eigenvector for $\lambda = -4$

Choose $c = 1 \Rightarrow \boxed{\vec{v}_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}}$

$\lambda_3 = 3$

$$[A-3I \mid \vec{0}] = \left[\begin{array}{ccc|c} -2 & 2 & 1 & 0 \\ 6 & -4 & 0 & 0 \\ -1 & -2 & -4 & 0 \end{array}\right] \xrightarrow[\substack{R_{3-1} \\ -R_3}]{-R_3} \left[\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 6 & -4 & 0 & 0 \\ 2 & -2 & -1 & 0 \end{array}\right] \xrightarrow[-2R_1+R_3]{-6R_1+R_2} \left[\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 0 & -16 & -24 & 0 \\ 0 & -6 & -9 & 0 \end{array}\right]$$

$$\xrightarrow[\substack{-\frac{1}{16}R_2 \\ -\frac{1}{9}R_3}]{} \left[\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 0 & 1 & \frac{3}{2} & 0 \\ 0 & 1 & \frac{3}{2} & 0 \end{array}\right] \xrightarrow[R_3-R_2]{} \left[\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 0 & 1 & \frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[R_1-2R_2]{} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & \frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow$ any vector of the form $\begin{bmatrix} -c \\ -\frac{3}{2}c \\ c \end{bmatrix}$ is an eigenvector for $\lambda = 3$

Choose $c = -2 \Rightarrow \boxed{\vec{v}_3 = \begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix}}$

Homogeneous Linear Systems with Constant Coefficients

Consider a homogeneous linear n-dimensional system:

$$\vec{x}' = A\vec{x}$$

where A is an n×n matrix of constants. Look for a solution of the form

$$\vec{x} = \vec{v}e^{\lambda t} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} e^{\lambda t}$$

where $\vec{v}$ is a vector of constants. If $\vec{x} = \vec{v}e^{\lambda t}$, then $\vec{x}' = \lambda \vec{v} e^{\lambda t}$, so

$$A\vec{x} - \vec{x}' = \vec{0} \iff A\vec{v}e^{\lambda t} - \lambda\vec{v}e^{\lambda t} = \vec{0} \iff (A\vec{v} - \lambda\vec{v})e^{\lambda t} = \vec{0}$$

$$\iff \boxed{A\vec{v} = \lambda\vec{v}}$$

$\vec{v}$ is an eigenvector of A corresponding to the eigenvalue λ.

Recall: When an n×n matrix has n distinct eigenvalues $\lambda_1, \dots, \lambda_n$, then a set of n linearly independent eigenvectors $\vec{v_1}, \dots, \vec{v_n}$ can always be found.

In this case, $\vec{x_1} = \vec{v_1}e^{\lambda_1 t}$; $\vec{x_2} = \vec{v_2}e^{\lambda_2 t}$; ... ; $\vec{x_n} = \vec{v_n}e^{\lambda_n t}$

are n linearly independent solutions to the system => fundamental set.

=> General Solution to Homogeneous Systems:

Case 1: The matrix A has n distinct real eigenvalues

Consider the homogeneous linear system of dimension n with constant coefficients: $\vec{x}' = A\vec{x}$.

If the matrix A has n distinct real eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ with corresponding eigenvectors $\vec{v_1}, \vec{v_2}, \dots, \vec{v_n}$, then the general solution to the system on $\mathbb{R}$ is:

$$\boxed{\vec{x} = C_1\vec{v_1}e^{\lambda_1 t} + C_2\vec{v_2}e^{\lambda_2 t} + \dots + C_n\vec{v_n}e^{\lambda_n t}}$$

Example: $\begin{cases} \dfrac{dx}{dt} = 2x + 3y \\[2mm] \dfrac{dy}{dt} = 2x + y \end{cases}$ $\qquad A = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}$

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 3 \\ 2 & 1-\lambda \end{vmatrix} = (2-\lambda)(1-\lambda) - 6 = \lambda^2 - 3\lambda - 4 = (\lambda-4)(\lambda+1)$$

Eigenvalues: $\boxed{\lambda \in \{4, -1\}}$

$\lambda = 4$: $[A - 4I \mid \vec{0}] = \left[\begin{array}{cc|c} -2 & 3 & 0 \\ 2 & -3 & 0 \end{array}\right] \xrightarrow{-R_1} \left[\begin{array}{cc|c} 2 & -3 & 0 \\ 2 & -3 & 0 \end{array}\right] \xrightarrow{R_2 - R_1} \left[\begin{array}{cc|c} 2 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$\Rightarrow 2V_1 - 3V_2 = 0 \Rightarrow V_1 = \frac{3}{2} V_2 \Rightarrow \vec{V_1} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$\lambda = -1$: $[A + I \mid \vec{0}] = \left[\begin{array}{cc|c} 3 & 3 & 0 \\ 2 & 2 & 0 \end{array}\right] \xrightarrow[\frac{1}{2}R_2]{\frac{1}{3}R_1} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right] \xrightarrow{R_2 - R_1} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$\Rightarrow V_1 + V_2 = 0 \Rightarrow V_1 = -V_2 \Rightarrow \vec{V_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\Rightarrow$ General Solution:

$$\boxed{\vec{x} = c_1 \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{4t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}}$$

Complex Eigenvalues

Consider a homogeneous linear system $\vec{x}' = A\vec{x}$ with constant coefficients. Suppose that

$$\boxed{\lambda = \alpha + i\beta} \quad \& \quad \boxed{\bar{\lambda} = \alpha - i\beta}$$

is a conjugate pair of eigenvalues of the matrix A. Then their corresponding eigenvectors also occur in conjugate pairs. That is, suppose that

$$\boxed{\vec{v} = \vec{a} + i\vec{b}}$$

is an eigenvector corresponding to λ. Then $\bar{\vec{v}}$ is an eigenvector for $\bar{\lambda}$. Therefore

$$\vec{u}(t) = \boxed{e^{\lambda t}\vec{v}} \quad \text{and} \quad \boxed{e^{\bar{\lambda} t}\bar{\vec{v}}} = \bar{\vec{u}}(t)$$

are solutions to the system. In terms of real-valued functions:

$$\boxed{\begin{aligned} \vec{x}_1(t) &= \operatorname{Re}\vec{u}(t) = e^{\alpha t}\left(\vec{a}\cos(\beta t) - \vec{b}\sin(\beta t)\right) \\ \vec{x}_2(t) &= \operatorname{Im}\vec{u}(t) = e^{\alpha t}\left(\vec{a}\sin(\beta t) + \vec{b}\cos(\beta t)\right) \end{aligned}}$$

are <u>real-valued</u>, <u>linearly independent</u> solutions.

$$\vec{u}(t) = e^{\lambda t}(\vec{a} + i\vec{b}) = e^{\alpha t}e^{i\beta t}(\vec{a} + i\vec{b}) = e^{\alpha t}\left(\cos(\beta t) + i\sin(\beta t)\right)(\vec{a} + i\vec{b})$$

$$= e^{\alpha t}\left[\left(\vec{a}\cos(\beta t) - \vec{b}\sin(\beta t)\right) + i\left(\vec{a}\sin(\beta t) + \vec{b}\cos(\beta t)\right)\right]$$

$$\Rightarrow \operatorname{Re}\vec{u}(t) = e^{\alpha t}\left(\vec{a}\cos(\beta t) - \vec{b}\sin(\beta t)\right) = \frac{1}{2}\left(\vec{u}(t) + \bar{\vec{u}}(t)\right)$$

$$\Rightarrow \operatorname{Im}\vec{u}(t) = e^{\alpha t}\left(\vec{a}\sin(\beta t) + \vec{b}\cos(\beta t)\right) = \frac{i}{2}\left(-\vec{u}(t) + \bar{\vec{u}}(t)\right)$$

both solutions to the system

$\Rightarrow$ If a pair $\lambda, \bar{\lambda}$ of complex eigenvalues occurs, the general solution must contain the terms

$$c_1\vec{x}_1(t) \quad \text{and} \quad c_2\vec{x}_2(t)$$

where $\vec{x}_{1,2}(t)$ are as above.

Example: $\vec{x}' = \begin{pmatrix} 2 & 8 \\ -1 & -2 \end{pmatrix} \vec{x}$

$$A = \begin{pmatrix} 2 & 8 \\ -1 & -2 \end{pmatrix} \Rightarrow \det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 8 \\ -1 & -2-\lambda \end{vmatrix} = -(4-\lambda^2) + 8 = \lambda^2 + 4$$

$$\Rightarrow \boxed{\lambda = \pm 2i}$$

Eigenvector for $\lambda = 2i$:

$$[A - 2iI \mid \vec{0}] = \left[\begin{array}{cc|c} 2-2i & 8 & 0 \\ -1 & -2-2i & 0 \end{array}\right]$$

$$\begin{cases} (2-2i)v_1 + 8v_2 = 0 \\ -v_1 - (2+2i)v_2 = 0 \end{cases} \Rightarrow \boxed{v_1 = -(2+2i)v_2} \qquad \Rightarrow \boxed{\vec{v} = \begin{bmatrix} 2+2i \\ -1 \end{bmatrix}}$$

$$\Rightarrow (2-2i)(2+2i)v_2 - 8v_2 = 0$$

$$\Rightarrow (4+4)v_2 - 8v_2 = 0 \Rightarrow 8v_2 - 8v_2 = 0 \checkmark$$

$$\Rightarrow \boxed{\operatorname{Re}\vec{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}; \ \operatorname{Im}\vec{v} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}}$$

$\Rightarrow$ General Solution:

$$\boxed{\vec{x} = c_1\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}\cos(2t) - \begin{bmatrix} 2 \\ 0 \end{bmatrix}\sin(2t)\right) + c_2\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}\sin(2t) + \begin{bmatrix} 2 \\ 0 \end{bmatrix}\cos(2t)\right)}$$

$$\boxed{= c_1\begin{bmatrix} 2\cos(2t) - 2\sin(2t) \\ -\cos(2t) \end{bmatrix} + c_2\begin{bmatrix} 2\sin(2t) + 2\cos(2t) \\ -\sin(2t) \end{bmatrix}}$$

Homogeneous Linear Systems with Constant Coefficients: Repeated Eigenvalues

Let λ be an eigenvalue of a matrix A:

- The **algebraic multiplicity** of λ is the multiplicity of λ as a solution to the characteristic equation of A.
- The **geometric multiplicity** of λ is the dimension of its eigenspace (the number of linearly independent eigenvectors λ can have).

If $\text{alg.}(\lambda) = \text{geom.}(\lambda)$, the eigenvalue λ is said to be **nondefective**.
Otherwise, if $\text{geom.}(\lambda) < \text{alg.}(\lambda)$, we say λ is **defective**.
A matrix which has a defective eigenvalue is called a **defective matrix**.

The 2×2 Case:

Example 1: $\vec{x}' = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \vec{x}$

Characteristic Equation: $\begin{vmatrix} 2-\lambda & 0 \\ 0 & 2-\lambda \end{vmatrix} = 0 \Rightarrow (2-\lambda)^2 = 0 \Rightarrow \boxed{\lambda = 2}$ (algebraic multiplicity = 2.)

Eigenvectors?

$[A - 2I \mid \vec{0}] = \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow$ No restrictions on the solutions!

$\Rightarrow$ Any $\vec{v} \in \mathbb{R}^2$ is an eigenvector! (geometric multiplicity = 2)

Choose for example $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and we have two linearly independent vectors.

$\Rightarrow$ General Solution: $\vec{x} = c_1 e^{2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\boxed{\vec{x} = e^{2t} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}}$

($\lambda = 2$ is nondefective).

Example 2: $\vec{x}' = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \vec{x}$

Characteristic Equation: $\begin{vmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{vmatrix} = 0 \Rightarrow (2-\lambda)^2 = 0 \Rightarrow \boxed{\lambda = 2}$ (algebraic multiplicity $=2$)

Eigenvectors?

$[A - 2I \mid \vec{0}] = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow$ One restriction: $v_2 = 0$

$\Rightarrow$ Eigenvector: $\boxed{\vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}}$ (geometric multiplicity $=1$)

$\Rightarrow \lambda = 2$ is a defective eigenvalue.

We only have $c_1 e^{2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ to put in the general solution.
How do we find a second solution?

Suppose A is a 2×2 matrix and λ is an eigenvalue of algebraic multiplicity 2 and geometric multiplicity 1.
Solve $\vec{x}' = A\vec{x}$:

1. Find an eigenvector $\vec{v}$ corresponding to λ. $\Rightarrow \vec{x}_1 = e^{\lambda t}\vec{v}$
2. Second Solution: a solution of the form:

$$\boxed{\vec{x}_2 = t e^{\lambda t}\vec{v} + e^{\lambda t}\vec{w}}$$

may be found, where $\vec{v}$ is the eigenvector in 1, and $\vec{w}$ is a vector that satisfies:

$$\boxed{(A - \lambda I)\vec{w} = \vec{v}}$$

General Solution:

$$\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2$$

Proof: Consider a vector $\vec{x}_2 = t e^{\lambda t}\vec{v} + e^{\lambda t}\vec{w}$. Then

$$\vec{x}_2' = A\vec{x}_2 \iff e^{\lambda t}\vec{v} + \lambda t e^{\lambda t}\vec{v} + \lambda e^{\lambda t}\vec{w} = t e^{\lambda t}\underbrace{A\vec{v}}_{=\lambda\vec{v}} + e^{\lambda t}A\vec{w}$$

$$\iff e^{\lambda t}\vec{v} + \cancel{\lambda t e^{\lambda t}\vec{v}} + \lambda e^{\lambda t}\vec{w} = \cancel{\lambda t e^{\lambda t}\vec{v}} + e^{\lambda t}A\vec{w}$$

$$\iff e^{\lambda t}\vec{v} = e^{\lambda t}A\vec{w} - \lambda e^{\lambda t}\vec{w}$$

$$\iff \vec{v} = A\vec{w} - \lambda\vec{w}$$

$$\iff \vec{v} = (A - \lambda I)\vec{w}.$$

Example 2 finished: $\vec{x}' = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \vec{x}$ $\boxed{\lambda = 2}$

Found: $\boxed{\vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}}$ $\Rightarrow \vec{x}_1 = e^{2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Second Solution: $\vec{x}_2 = te^{2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + e^{2t}\vec{w}$

Find $\vec{w}$: $(A - 2I)\vec{w} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \vec{w} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} w_2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow w_2 = 1$$

$\Rightarrow \vec{w}$ can be any vector of the form $\begin{pmatrix} c \\ 1 \end{pmatrix}$ with $c \in \mathbb{C}$.

Choose $c = 0$: $\vec{w} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\Rightarrow \vec{x}_2 = te^{2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + e^{2t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

$$\Rightarrow \vec{x} = c_1 e^{2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2\left[te^{2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + e^{2t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]$$

$$\vec{x} = e^{2t}\left[\begin{pmatrix} c_1 \\ 0 \end{pmatrix} + \begin{pmatrix} c_2 t \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ c_2 \end{pmatrix}\right] \Rightarrow \boxed{\vec{x} = e^{2t}\begin{pmatrix} c_1 + c_2 t \\ c_2 \end{pmatrix}}$$

Non-Homogeneous Linear Systems : Variation of Parameters

Recall : For a standard form linear equation $y'' + P(x)y' + Q(x)y = g(x)$ we obtained y_p from $y_c = c_1 y_1 + c_2 y_2$ by "varying" c_1, c_2 and looking for

$$y_p = u_1(t)y_1(t) + u_2(t)y_2(t).$$

Same idea for a linear system :

- Consider the nonhomogeneous linear system : $\boxed{\vec{x}' = A\vec{x} + \vec{g}(t)}$
- Complementary Solution : Recall that the solution to $\vec{x}' = A\vec{x}$ can be expressed as : $\boxed{\vec{x}_c = \Phi(t)\vec{c}}$ where $\Phi(t)$ is a fundamental matrix and

$$\vec{c} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$ is a vector of arbitrary constants.

- Particular Solution : Replace $\vec{c}$ by $\vec{u}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_n(t) \end{pmatrix}$ so that $\boxed{\vec{x}_p = \Phi(t)\vec{u}(t)}$ is a particular solution to $\vec{x}' = A\vec{x} + \vec{g}(t)$.

$$\vec{x}_p = \Phi(t)\vec{u}(t) \Rightarrow \vec{x}_p' = \Phi'(t)\vec{u}(t) + \Phi(t)\vec{u}'(t)$$

$$\vec{x}_p' = A\vec{x}_p + \vec{g}(t) \Rightarrow \Phi'(t)\vec{u}(t) + \Phi(t)\vec{u}'(t) = A\Phi(t)\vec{u}(t) + \vec{g}(t)$$

$$\boxed{\Phi(t)' = A\Phi(t)} \Rightarrow \cancel{A\Phi(t)\vec{u}(t)} + \Phi(t)\vec{u}'(t) = \cancel{A\Phi(t)\vec{u}(t)} + \vec{g}(t)$$

$$\Rightarrow \Phi(t)\vec{u}'(t) = \vec{g}(t)$$

$$\Rightarrow \boxed{\vec{u}'(t) = \Phi^{-1}(t)\vec{g}(t)}$$

$$\Rightarrow \boxed{\vec{u}(t) = \int \Phi^{-1}(t)\vec{g}(t)\,dt}$$

$$\Rightarrow \boxed{\vec{x}_p = \Phi(t)\int \Phi^{-1}(t)\vec{g}(t)\,dt}$$

Why? $\vec{x}_c = \Phi(t)\vec{c}$ is a solution to $\vec{x}' = A\vec{x}$, so

$$\Phi'(t)\vec{c} = A\Phi(t)\vec{c}$$

$$[\Phi'(t) - A\Phi(t)]\vec{c} = \vec{0}$$

Since this is true for all $\vec{c} \in \mathbb{R}^n$, we must have

$$\Phi'(t) = A\Phi(t)$$

- General Solution : $\boxed{\vec{x} = \vec{x}_c + \vec{x}_p = \Phi(t)\vec{c} + \Phi(t)\int \Phi^{-1}(t)\vec{g}(t)\,dt}$

Example: $\vec{x}' = \begin{pmatrix} -3 & 1 \\ 2 & -4 \end{pmatrix}\vec{x} + \begin{pmatrix} 3t \\ e^{-t} \end{pmatrix}$

- Complementary Sol. $\vec{x}' = A\vec{x}$

$$\begin{vmatrix} -3-\lambda & 1 \\ 2 & -4-\lambda \end{vmatrix} = (\lambda+3)(\lambda+4)-2 = \lambda^2+7\lambda+10 = (\lambda+2)(\lambda+5)$$

Eigenvalues: $\lambda_1=-2$; $\lambda_2=-5$

$$[A+2I|\vec{0}] = \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 2 & -2 & 0 \end{array}\right] \Rightarrow v_1=v_2 \Rightarrow \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \vec{x}_1 = e^{-2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$[A+5I|\vec{0}] = \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 2 & 1 & 0 \end{array}\right] \Rightarrow v_2 = -2v_1 \Rightarrow \vec{v}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \Rightarrow \vec{x}_2 = e^{-5t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$\Rightarrow \boxed{\Phi(t) = \begin{pmatrix} e^{-2t} & e^{-5t} \\ e^{-2t} & -2e^{-5t} \end{pmatrix}} \Rightarrow \det(\Phi(t)) = -3e^{-7t}$$

- Particular Solution:

$$\Phi^{-1}(t) = \frac{1}{-3e^{-7t}}\begin{pmatrix} -2e^{-5t} & -e^{-5t} \\ -e^{-2t} & e^{-2t} \end{pmatrix} = \begin{pmatrix} \frac{2}{3}e^{2t} & \frac{1}{3}e^{2t} \\ \frac{1}{3}e^{5t} & -\frac{1}{3}e^{5t} \end{pmatrix}$$

$$\Phi^{-1}(t)\vec{g}(t) = \begin{pmatrix} \frac{2}{3}e^{2t} & \frac{1}{3}e^{2t} \\ \frac{1}{3}e^{5t} & -\frac{1}{3}e^{5t} \end{pmatrix}\begin{pmatrix} 3t \\ e^{-t} \end{pmatrix} = \begin{pmatrix} 2te^{2t}+\frac{1}{3}e^{t} \\ te^{5t}-\frac{1}{3}e^{4t} \end{pmatrix}$$

$$\Rightarrow \vec{u}(t) = \int \begin{pmatrix} 2te^{2t}+\frac{1}{3}e^{t} \\ te^{5t}-\frac{1}{3}e^{4t} \end{pmatrix} dt = \begin{pmatrix} te^{2t}-\frac{1}{2}e^{2t}+\frac{1}{3}e^{t} \\ \frac{1}{5}te^{5t}-\frac{1}{25}e^{5t}-\frac{1}{12}e^{4t} \end{pmatrix}$$

$$\Rightarrow \vec{x}_p = \Phi(t)\vec{u}(t) = \begin{pmatrix} e^{-2t} & e^{-5t} \\ e^{-2t} & -2e^{-5t} \end{pmatrix}\begin{pmatrix} te^{2t}-\frac{1}{2}e^{2t}+\frac{1}{3}e^{t} \\ \frac{1}{5}te^{5t}-\frac{1}{25}e^{5t}-\frac{1}{12}e^{4t} \end{pmatrix}$$

$$= \begin{pmatrix} t-\frac{1}{2}+\frac{1}{3}e^{-t}+\frac{1}{5}t-\frac{1}{25}-\frac{1}{12}e^{-t} \\ t-\frac{1}{2}+\frac{1}{3}e^{-t}-\frac{2}{5}t+\frac{2}{25}+\frac{1}{6}e^{-t} \end{pmatrix} = \begin{pmatrix} \frac{6}{5}t+\frac{1}{4}e^{-t}-\frac{27}{50} \\ \frac{3}{5}t+\frac{1}{2}e^{-t}-\frac{21}{50} \end{pmatrix}$$

Recall: Inverting a matrix using Elementary Row Operations

$$A = \begin{pmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{pmatrix} \quad ; \quad A^{-1} = ?$$

$$\left(\begin{array}{ccc|ccc} 3 & 0 & 2 & 1 & 0 & 0 \\ 2 & 0 & -2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow[\frac{1}{2}R_2]{\frac{1}{3}R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 2/3 & 1/3 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1/2 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[R_2-R_1]{} \left(\begin{array}{ccc|ccc} 1 & 0 & 2/3 & 1/3 & 0 & 0 \\ 0 & 0 & -5/3 & -1/3 & 1/2 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow[-3/5\,R_2]{} \left(\begin{array}{ccc|ccc} 1 & 0 & 2/3 & 1/3 & 0 & 0 \\ 0 & 0 & 1 & 1/5 & -3/10 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[R_3-R_2]{R_1-2/3R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/5 & 1/5 & 0 \\ 0 & 0 & 1 & 1/5 & -3/10 & 0 \\ 0 & 1 & 0 & -1/5 & 3/10 & 1 \end{array}\right) \xrightarrow[R_2 \leftrightarrow R_3]{} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/5 & 1/5 & 0 \\ 0 & 1 & 0 & -1/5 & 3/10 & 1 \\ 0 & 0 & 1 & 1/5 & -3/10 & 0 \end{array}\right)$$

$$\Rightarrow \boxed{A^{-1} = \begin{pmatrix} 1/5 & 1/5 & 0 \\ -1/5 & 3/10 & 1 \\ 1/5 & -3/10 & 0 \end{pmatrix}}$$

Matrix Exponential

- Recall: if $x(t)$ is a function of t, the equation $x' = ax$ has general solution $x = ce^{at}$.
- Question: Can we somehow extend this to homogeneous linear systems?

That is, can we have the general solution of $\vec{x}' = A\vec{x}$ be of the form $\vec{x} = e^{tA}\vec{c}$?

Definition: Let A be an $n \times n$ matrix. Define the matrix exponential:

$$\boxed{e^{tA} := \sum_{n=0}^{\infty} \frac{(tA)^n}{n!} = I + tA + \frac{t^2}{2} A^2 + \frac{t^3}{3!} A^3 + \ldots} \qquad (*)$$

- The series in $(*)$ converges to an $n \times n$ matrix for all real t.
- The matrix e^{tA} is invertible, with inverse e^{-tA}.
- Take $t=0$ in $(*)$: $\boxed{e^{0 \cdot A} = I}$
- Differentiate both sides of $(*)$: $\boxed{\frac{d}{dt} e^{tA} = Ae^{tA}}$

$$\frac{d}{dt} e^{tA} = A + tA^2 + \frac{t^2}{2!} A^3 + \ldots = A\left(I + tA + \frac{t^2}{2!} A^2 + \ldots\right) = Ae^{tA}.$$

$\Rightarrow$ e^{tA} is the special fundamental matrix $\Psi(t)$ of the system $\vec{x}' = A\vec{x}$ that satisfies $\Psi(0) = I$. $\boxed{e^{tA} = \Psi(t)}$

Recall: If $\Phi(t)$ is a fundamental matrix of $\vec{x}' = A\vec{x}$, then $\Phi(t)' = A\Phi(t)$. Conversely, if $\Phi(t)$ is an invertible matrix that satisfies $\Phi'(t) = A\Phi(t)$ for all t in some interval I, then $\Phi(t)$ is a fundamental matrix of $\vec{x}' = A\vec{x}$.

Let $\Psi(t) = e^{tA}$. Then $\boxed{\frac{d}{dt} e^{tA} = Ae^{tA}}$ translates to $\Psi'(t) = A\Psi(t)$. Thus e^{tA} is a fundamental matrix for the system $\vec{x}' = A\vec{x}$. Moreover, we know that $e^{0 \cdot A} = I$, or $\Psi(0) = I$. So $\Psi(t)$ is the (special) fundamental matrix that satisfies $\Psi(0) = I$.

Relationship to Variation of Parameters :

Recall that we obtained the general solution of $\vec{x}' = A\vec{x} + \vec{g}(t)$ to be

$$\vec{x}(t) = \Phi(t)\vec{c} + \Phi(t)\int \Phi^{-1}(t)\vec{g}(t)\,dt$$

where $\Phi(t)$ is a fundamental matrix of the homogeneous system $\vec{x}' = A\vec{x}$. So take $\Phi(t)$ to be $\Psi(t) = e^{tA}$ above:

$$\begin{aligned}\vec{x}(t) &= \Psi(t)\vec{c} + \Psi(t)\int \Psi^{-1}(t)\vec{g}(t)\,dt \\ &= e^{tA}\vec{c} + e^{tA}\int \underbrace{(e^{tA})^{-1}}\vec{g}(t)\,dt\end{aligned}$$

this is simply e^{-tA} !

$$\Rightarrow \boxed{\vec{x}(t) = e^{tA}\vec{c} + e^{tA}\int e^{-tA}\,\vec{g}(t)\,dt}$$ is the general solution to $\vec{x}' = A\vec{x} + \vec{g}(t)$

$\Rightarrow$ If you know e^{tA}, then you can find the solution to any non-homogeneous linear system without having to invert any matrices ! Because $(e^{tA})^{-1} = e^{-tA}$, so all you have to do is replace t by $(-t)$ in e^{-tA} and you have your inverse.

Example: Find e^{tA} for $A=\begin{pmatrix}-3 & 1\\ 2 & -4\end{pmatrix}$.

Solve $\vec{x}'=A\vec{x}$: we already did this earlier and found the fundamental matrix

$$\Phi(t)=\begin{pmatrix}e^{-2t} & e^{-5t}\\ e^{-2t} & -2e^{-5t}\end{pmatrix}.$$

Goal : Find $e^{tA}=\Psi(t)$, the fundamental matrix with $\Psi(0)=I$.

Two ways to do this:

I.) Use: $\boxed{\Psi(t)=\Phi(t)\Phi^{-1}(0)}$ (this does involve finding $\Phi^{-1}(t)$).

We already computed $\Phi^{-1}(t)=\frac{1}{3}\begin{pmatrix}2e^{2t} & e^{2t}\\ e^{5t} & -e^{5t}\end{pmatrix} \Rightarrow \Phi^{-1}(0)=\frac{1}{3}\begin{pmatrix}2 & 1\\ 1 & -1\end{pmatrix}$

$$\Rightarrow \Psi(t)=\begin{pmatrix}e^{-2t} & e^{-5t}\\ e^{-2t} & -2e^{-5t}\end{pmatrix}\frac{1}{3}\begin{pmatrix}2 & 1\\ 1 & -1\end{pmatrix}$$

$$e^{tA}=\Psi(t)=\frac{1}{3}\begin{pmatrix}2e^{-2t}+e^{-5t} & e^{-2t}-e^{-5t}\\ 2e^{-2t}-2e^{-5t} & e^{-2t}+2e^{-5t}\end{pmatrix}$$

II.) Solve $\boxed{\vec{x}=\Phi(t)\vec{c}}$ subject to $\boxed{\vec{x}(0)=\vec{e}_1\,;\ \vec{x}(0)=\vec{e}_2;\ \dots\ \vec{x}(0)=\vec{e}_n}$ and obtain the n columns of $\Psi(t)$ (this does not involve inverting matrices)

$$\vec{x}(t)=\Phi(t)\vec{c}\Rightarrow \vec{x}(0)=\Phi(0)\vec{c}=\begin{pmatrix}1 & 1\\ 1 & -2\end{pmatrix}\begin{pmatrix}c_1\\ c_2\end{pmatrix}=\begin{pmatrix}c_1+c_2\\ c_1-2c_2\end{pmatrix}$$

⊛ $\begin{pmatrix}c_1+c_2\\ c_1-2c_2\end{pmatrix}=\begin{pmatrix}1\\ 0\end{pmatrix}\Rightarrow\begin{cases}c_1+c_2=1\\ c_1-2c_2=0\end{cases}$ $\begin{matrix}3c_2=1\Rightarrow c_2=1/3\\ c_1=2c_2\Rightarrow c_1=2/3\end{matrix}$ $\Rightarrow \vec{x}_1=\Phi(t)\begin{pmatrix}2/3\\ 1/3\end{pmatrix}$

$$\Rightarrow \vec{x}_1=\frac{1}{3}\begin{pmatrix}e^{-2t} & e^{-5t}\\ e^{-2t} & -2e^{-5t}\end{pmatrix}\begin{pmatrix}2\\ 1\end{pmatrix}\Rightarrow \vec{x}_1=\frac{1}{3}\begin{pmatrix}2e^{-2t}+e^{-5t}\\ 2e^{-2t}-2e^{-5t}\end{pmatrix}\leftarrow$$

First column of $\Psi(t)$ ✓

⊛ $\begin{pmatrix}c_1+c_2\\ c_1-2c_2\end{pmatrix}=\begin{pmatrix}0\\ 1\end{pmatrix}\Rightarrow\begin{cases}c_1+c_2=0\\ c_1-2c_2=1\end{cases}$ $\begin{cases}c_1=-c_2\Rightarrow c_1=1/3\\ -3c_2=1\Rightarrow c_2=-1/3\end{cases}$ $\Rightarrow \vec{x}_2=\Phi(t)\begin{pmatrix}1/3\\ -1/3\end{pmatrix}$

$$\Rightarrow \vec{x}_2=\frac{1}{3}\begin{pmatrix}e^{-2t} & e^{-5t}\\ e^{-2t} & -2e^{-5t}\end{pmatrix}\begin{pmatrix}1\\ -1\end{pmatrix}\Rightarrow \vec{x}_2=\frac{1}{3}\begin{pmatrix}e^{-2t}-e^{-5t}\\ e^{-2t}+2e^{-5t}\end{pmatrix}\leftarrow$$

Second column of $\Psi(t)$ ✓

Find $e^{-tA} = (e^{tA})^{-1}$

$$e^{tA} = \frac{1}{3}\begin{pmatrix} 2e^{-2t}+e^{-5t} & e^{-2t}-e^{-5t} \\ 2e^{-2t}-2e^{-5t} & e^{-2t}+2e^{-5t} \end{pmatrix} \Rightarrow e^{-tA} = \frac{1}{3}\begin{pmatrix} 2e^{2t}+e^{5t} & e^{2t}-e^{5t} \\ 2e^{2t}-2e^{5t} & e^{2t}+2e^{5t} \end{pmatrix}$$

Use the matrix exponential to solve:

$$\vec{x}' = \begin{pmatrix} -3 & 1 \\ 2 & -4 \end{pmatrix}\vec{x} + \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

General Solution: $\boxed{\vec{x}(t) = e^{tA}\vec{c} + e^{tA}\int e^{-tA}\vec{g}(t)\,dt}$ $\quad \vec{g}(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$e^{-tA}\vec{g}(t) = \frac{1}{3}\begin{pmatrix} e^{2t}+2e^{5t} \\ e^{2t}-4e^{5t} \end{pmatrix} \Rightarrow \int e^{-tA}\vec{g}(t)\,dt = \frac{1}{3}\begin{pmatrix} \frac{1}{2}e^{2t}+\frac{2}{5}e^{5t} \\ \frac{1}{2}e^{2t}-\frac{4}{5}e^{5t} \end{pmatrix}$$

$$\Rightarrow \vec{x}_p = e^{tA}\int e^{-tA}\vec{g}(t)\,dt = \frac{1}{3}e^{-5t}\begin{pmatrix} 2e^{3t}+1 & e^{3t}-1 \\ 2e^{3t}-2 & e^{3t}+2 \end{pmatrix}\left[\frac{1}{6}e^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{2}{15}e^{5t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}\right]$$

$$= \frac{1}{18}e^{-3t}\begin{pmatrix} 3e^{3t} \\ 3e^{3t} \end{pmatrix} + \frac{2}{45}\begin{pmatrix} 3 \\ -6 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{2}{15}\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 3/10 \\ -1/10 \end{pmatrix}$$

$$\Rightarrow \boxed{\vec{x}_p = \frac{1}{10}\begin{pmatrix} 3 \\ -1 \end{pmatrix}}$$

Check: $\vec{x}_p{}' = \vec{0}$

$$A\vec{x}_p + \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -3 & 1 \\ 2 & -4 \end{pmatrix}\frac{1}{10}\begin{pmatrix} 3 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{10}\begin{pmatrix} -10 \\ 10 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \vec{0}$$

$$\Rightarrow A\vec{x}_p + \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \vec{x}_p{}'. \quad //$$

$$\Rightarrow \boxed{\vec{x}(t) = e^{tA}\vec{c} + \frac{1}{10}\begin{pmatrix} 3 \\ -1 \end{pmatrix}}$$