The Laplace Transform

$$\boxed{\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt.}$$

Laplace transforms of some basic functions

$\mathscr{L}\{1\} = \frac{1}{s};\ \ s > 0$

$\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}};\ \ s > 0$

$\mathscr{L}\{e^{kt}\} = \frac{1}{s-k};\ \ s > k$

$\mathscr{L}\{\sin(kt)\} = \frac{k}{s^2+k^2};\ \ s > 0$

$\mathscr{L}\{\cos(kt)\} = \frac{s}{s^2+k^2};\ \ s > 0$

$\mathscr{L}\{u_a(t)\} = \frac{e^{-as}}{s};\ s > 0$

$\mathscr{L}\{\sinh(kt)\} = \frac{k}{s^2-k^2};\ \ s > |k|$

$\mathscr{L}\{\cosh(kt)\} = \frac{s}{s^2-k^2};\ \ s > |k|$

Inverse Laplace transforms of some basic functions

$\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} = 1$

$\mathscr{L}^{-1}\left\{\frac{1}{s^n}\right\} = \frac{1}{(n-1)!}t^{n-1}$

$\mathscr{L}^{-1}\left\{\frac{1}{s-k}\right\} = e^{kt}$

$\mathscr{L}^{-1}\left\{\frac{1}{s^2+k^2}\right\} = \frac{1}{k}\sin(kt)$

$\mathscr{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\} = \cos(kt)$

$\mathscr{L}^{-1}\left\{\frac{e^{-as}}{s}\right\} = u_a(t)$

$\mathscr{L}^{-1}\left\{\frac{1}{s^2-k^2}\right\} = \frac{1}{k}\sinh(kt)$

$\mathscr{L}^{-1}\left\{\frac{s}{s^2-k^2}\right\} = \cosh(kt)$

Properties of the Laplace and Inverse Laplace transform

Translation Theorem I:

$\mathscr{L}\left\{e^{kt}f(t)\right\} = F(s-k) = \mathscr{L}\{f(t)\}|_{s\to s-k}$

$\mathscr{L}^{-1}\{F(s-k)\} = \mathscr{L}^{-1}\left\{F(s)|_{s\to s-k}\right\} = e^{kt}\mathscr{L}^{-1}\{F(s)\} = e^{kt}f(t)$

Translation Theorem II:

$\mathscr{L}\left\{f(t-a)u_a(t)\right\} = e^{-as}F(s) = e^{-as}\mathscr{L}\{f(t)\}$

$\mathscr{L}^{-1}\{e^{-as}F(s)\} = f(t-a)u_a(t) = \mathscr{L}^{-1}\{F(s)\}\big|_{t\to t-a}\, u_a(t)$

Derivatives of Laplace Transforms:

$\mathscr{L}\left\{t^n f(t)\right\} = (-1)^n \frac{d^n}{ds^n}F(s)$

$\mathscr{L}^{-1}\left\{F^{(n)}(s)\right\} = (-1)^n t^n f(t)$

Laplace Transform of Periodic Functions:
If $f(t)$ is piecewise continuous on $[0,\infty)$, of exponential order, and periodic with period T:

$$\mathscr{L}\{f(t)\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)\, dt$$

Laplace Transforms of Derivatives:

$$\begin{aligned}
\mathscr{L}\{y'\} &= sY(s) - y(0)\\
\mathscr{L}\{y''\} &= s^2Y(s) - sy(0) - y'(0)\\
\mathscr{L}\{y'''\} &= s^3Y(s) - s^2y(0) - sy'(0) - y''(0)\\
&\ \ \vdots\\
\mathscr{L}\{y^{(n)}(t)\} &= s^nY(s) - s^{n-1}y(0) - s^{n-2}y'(0) - \ldots - sy^{(n-2)}(0) - y^{(n-1)}(0)
\end{aligned}$$

Inverse Laplace Transform

- Laplace transform: $\boxed{f(t)} \xrightarrow{\mathcal{L}} \boxed{F(s) = \mathcal{L}\{f(t)\}}$
- Inverse problem: given $F(s)$, find $f(t)$ such that $F(s) = \mathcal{L}\{f(t)\} \Rightarrow \boxed{f(t) = \mathcal{L}^{-1}\{F(s)\}}$

$$\boxed{f(t) = \mathcal{L}^{-1}\{F(s)\}} \xleftarrow{\mathcal{L}^{-1}} \boxed{F(s)}$$

Inverse Laplace Transforms we already know:

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1 \qquad \mathcal{L}^{-1}\left\{\frac{1}{s^2+k^2}\right\} = \frac{1}{k}\sin(kt) \qquad \mathcal{L}^{-1}\left\{\frac{1}{s^2-k^2}\right\} = \frac{1}{k}\sinh(kt)$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^n}\right\} = \frac{1}{(n-1)!}t^{n-1} \qquad \mathcal{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\} = \cos(kt) \qquad \mathcal{L}^{-1}\left\{\frac{s}{s^2-k^2}\right\} = \cosh(kt)$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s-k}\right\} = e^{kt}$$

Basic Examples:

$$\mathcal{L}^{-1}\left\{\frac{1}{s^5}\right\} = \frac{1}{4!}t^4 \quad ; \quad \mathcal{L}^{-1}\left\{\frac{3}{s^2+64}\right\} = \frac{3}{8}\sin(8t) \quad ; \quad \mathcal{L}^{-1}\left\{\frac{5}{s^2-16}\right\} = \frac{5}{4}\sinh(4t).$$

$$\mathcal{L}^{-1}\left\{\frac{3}{s}\right\} = 3 \quad ; \quad \mathcal{L}^{-1}\left\{\frac{2s}{s^2+64}\right\} = 2\cos(8t) \quad ; \quad \mathcal{L}^{-1}\left\{\frac{\sqrt{2}s}{s^2-7}\right\} = \sqrt{2}\cosh(\sqrt{7}t).$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} = e^{-2t} \quad ; \quad \mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} = e^{3t} \quad ; \quad \mathcal{L}^{-1}\left\{\frac{2s-1}{s^2+16}\right\} = 2\cos(4t) - \frac{1}{4}\sin(4t).$$

Translation Theorem (Inverse Laplace Form):

$$\boxed{\mathcal{L}^{-1}\{F(s-k)\} = \mathcal{L}^{-1}\{F(s)|_{s\to s-k}\} = e^{kt}f(t)} \; ; \; f(t) = \mathcal{L}^{-1}\{F(s)\}$$

Examples:

- $\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^3}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^3}\Big|_{s\to s+2}\right\} = e^{-2t}\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\} = \frac{1}{2}e^{-2t}t^2.$

- $\mathcal{L}^{-1}\left\{\frac{s}{s^2+6s+11}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{(s+3)^2+2}\right\} = \mathcal{L}^{-1}\left\{\frac{s+3-3}{(s+3)^2+2}\right\}$

[Completing the square]

$$= \mathcal{L}^{-1}\left\{\frac{s+3}{(s+3)^2+2}\right\} - 3\mathcal{L}^{-1}\left\{\frac{1}{(s+3)^2+2}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s}{s^2+2}\Big|_{s\to s+3}\right\} - 3\mathcal{L}^{-1}\left\{\frac{1}{s^2+2}\Big|_{s\to s+3}\right\}$$

$$= e^{-3t}\cos(\sqrt{2}t) - \frac{3}{\sqrt{2}}e^{-3t}\sin(\sqrt{2}t).$$

Behavior of $F(s)$ as $s\to\infty$:

If $f(t)$ is a piecewise continuous function of exponential order on $t\in[0,\infty)$, then $\boxed{\lim_{s\to\infty}\left[\mathcal{L}\{f(t)\}\right] = 0}$

$\Rightarrow$ The Laplace transforms of $F_1(s) = s^2$ and $F_2(s) = \frac{s}{s+1}$ do not exist b/c $\lim_{s\to\infty} F_1(s) = \infty$; $\lim_{s\to\infty} F_2(s) = 1.$

$\mathcal{L}^{-1}$ using Partial Fractions

Look at 3 cases:

- Distinct Linear Factors: $\dfrac{1}{(s-1)(s+2)(s+4)}$
- Repeated Linear Factors: $\dfrac{s+1}{s(s+2)^3}$
- Irreducible Quadratic Factors: $\dfrac{3s-2}{s(s^2+4)}$

1. Distinct Linear Factors: $F(s) = \dfrac{1}{(s-1)(s+2)(s+4)}$

$$F(s) = \frac{1}{(s-1)(s+2)(s+4)} = \frac{A}{s-1} + \frac{B}{s+2} + \frac{C}{s+4}$$

$$\Rightarrow \mathcal{L}^{-1}\{F(s)\} = \underset{?}{A}\,\underbrace{\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}}_{e^{t}} + \underset{?}{B}\,\underbrace{\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}}_{e^{-2t}} + \underset{?}{C}\,\underbrace{\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\}}_{e^{-4t}}$$

$$\boxed{\frac{1}{(s-1)(s+2)(s+4)} = \frac{A}{s-1} + \frac{B}{s+2} + \frac{C}{s+4}}$$

Bring to the same denominator:

$$1 = A(s+2)(s+4) + B(s-1)(s+4) + C(s-1)(s+2)$$

Give convenient values to s:

$s=1 \Rightarrow 1 = 15A \Rightarrow A = 1/15$

$s=-2 \Rightarrow 1 = -6B \Rightarrow B = -1/6$

$s=-4 \Rightarrow 1 = 10C \Rightarrow C = 1/10$

"Cover up" Method:

- Cover up the term $(s-1)$ and evaluate at $s=1$:

$$\left.\frac{1}{\boxed{(s-1)}(s+2)(s+4)}\right|_{s=1} = A \Rightarrow A = \frac{1}{15}$$

- Same for the other terms:

$$\left.\frac{1}{(s-1)\boxed{(s+2)}(s+4)}\right|_{s=-2} = B \Rightarrow B = \frac{-1}{6}$$

$$\left.\frac{1}{(s-1)(s+2)\boxed{(s+4)}}\right|_{s=-4} = C \Rightarrow C = \frac{1}{10}$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{(s-1)(s+2)(s+4)}\right\} = \frac{1}{15}e^{t} - \frac{1}{6}e^{-2t} + \frac{1}{10}e^{-4t}.$$

② Repeated Linear Factors: $F(s) = \dfrac{3s^2-16s+21}{(s-1)^2(s+3)}$

$$F(s) = \frac{3s^2-16s+21}{(s-1)^2(s+3)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s+3}$$

$$\Rightarrow \mathcal{L}^{-1}\{F(s)\} = \underset{?}{A}\,\underbrace{\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}}_{e^t} + \underset{?}{B}\,\underbrace{\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\}}_{e^t \cdot t} + \underset{?}{C}\,\underbrace{\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\}}_{e^{-3t}}$$

$$\boxed{\frac{3s^2-16s+21}{(s-1)^2(s+3)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s+3}}$$

Problem: When repeated factors occur, giving the "convenient" values to s does not give you all the coefficients.

$$3s^2-16s+21 = A(s-1)(s+3) + B(s+3) + C(s-1)^2$$

$s=1 \Rightarrow 8 = 4B \Rightarrow B=2$

$s=-3 \Rightarrow 96 = 16C \Rightarrow C=6$

$$\left.\frac{3s^2-16s+21}{\boxed{(s-1)^2}(s+3)}\right|_{s=1} = B \Rightarrow B = \frac{8}{4} = 2$$

$$\left.\frac{3s^2-16s+21}{(s-1)^2\boxed{(s+3)}}\right|_{s=-3} = C \Rightarrow C = \frac{96}{16} = 6.$$

To find A, you can for example:

- Match coefficients of a power of s:

  Ex: s^2 coefficients: $3 = A + C = A + 6 \Rightarrow A = -3$

  or s coefficients: $-16 = 2A + B - 2C = 2A - 10 \Rightarrow 2A = -6 \Rightarrow A = -3$

  or s^0 coefficients: $21 = -3A + 3B + C = -3A + 12 \Rightarrow -3A = 9 \Rightarrow A = -3.$

- Give some other value to s:

  Ex: $s=0 \Rightarrow 21 = -3A + 3B + C$ (essentially the same as matching the coefficients of s^0).

$$\mathcal{L}^{-1}\left\{\frac{3s^2-16s+21}{(s-1)^2(s+3)}\right\} = -3e^t + 2te^t + 6e^{-3t}.$$

③ Irreducible Quadratic Factor: $F(s) = \frac{3s}{(s^2+2)(s-1)}$

$$F(s) = \frac{3s}{(s^2+2)(s-1)} = \frac{As+B}{s^2+2} + \frac{C}{s-1}$$

$$\Rightarrow \mathcal{L}^{-1}\{F(s)\} = \underset{?}{A}\,\underbrace{\mathcal{L}^{-1}\left\{\frac{s}{s^2+2}\right\}}_{\cos(\sqrt{2}t)} + \underset{?}{B}\,\underbrace{\mathcal{L}^{-1}\left\{\frac{1}{s^2+2}\right\}}_{\frac{1}{\sqrt{2}}\sin(\sqrt{2}t)} + \underset{?}{C}\,\underbrace{\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}}_{e^t}$$

$$\boxed{\frac{3s}{(s^2+2)(s-1)} = \frac{As+B}{s^2+2} + \frac{C}{s-1}}$$

Problem: An irreducible factor has no "convenient" values to give s.

$$\boxed{3s = (As+B)(s-1) + C(s^2+2)}$$

Convenient value: $s=1 \Rightarrow 3 = 3C \Rightarrow C=1$

$$\left.\frac{3s}{(s^2+2)\boxed{(s-1)}}\right|_{s=1} = C \Rightarrow C = \frac{3}{3} = 1.$$

To find the remaining coefficients A & B, you can:

- Come up with a 2x2 linear equations in A & B. You can do this by
  - Matching coefficients of powers of s:
    - s^2 coefficients: $0 = A + C = A + 1 \Rightarrow A = -1$
    - s^0 coefficients: $0 = -B + 2C = -B + 2 \Rightarrow B = 2$
  - Giving other values to s:
    - $s=0 \Rightarrow 0 = -B + 2C \Rightarrow B = 2$
    - $s=2 \Rightarrow 6 = (2A+B) + 6C \Rightarrow 6 = 2A + 2 + 6 \Rightarrow A = -1.$
- Subtract the terms you know. That is, once you know $C = 1$:

$$\frac{3s}{(s^2+2)(s-1)} = \frac{As+B}{s^2+2} + \frac{1}{s-1}$$

$$\Rightarrow \frac{As+B}{s^2+2} = \frac{3s}{(s^2+2)(s-1)} - \frac{1}{s-1} = \frac{3s - s^2 - 2}{(s^2+2)(s-1)} = -\frac{s^2 - 3s + 2}{(s^2+2)(s-1)} = -\frac{(s-2)\cancel{(s-1)}}{(s^2+2)\cancel{(s-1)}}$$

This equality should tell you that $(s-1)$ has to be a factor of $(3s - s^2 - 2)$

$$\Rightarrow \frac{As+B}{s^2+2} = \frac{-s+2}{s^2+2}$$ $\Rightarrow$ You can just determine $\boxed{A=-1;\ B=2}$ by inspection at this point.

Laplace Transform & Linear ODEs w/ Constant Coefficients (IVP)

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = g(t)$$
$$y(0) = c_0;\ y'(0) = c_1; \dots;\ y^{(n-1)}(0) = c_{n-1}$$

$\xrightarrow[\text{of both sides}]{\mathcal{L}}$ Algebraic Equation: Solve for $Y(s) = \mathcal{L}\{y(t)\}$ $\xrightarrow{\mathcal{L}^{-1}}$ Solution $y(t)$

$$\mathcal{L}\{y'\} = sY(s) - y(0);\ \mathcal{L}\{y''\} = s^2Y(s) - sy(0) - y'(0);\ \mathcal{L}\{y'''\} = s^3Y(s) - s^2y(0) - sy'(0) - y''(0)$$

Example: $\boxed{y'' - 4y' + 4y = t^3e^{2t};\ y(0) = y'(0) = 0}$ Solved using all 3 methods so far.

Laplace Solution

$$\mathcal{L}\{y'' - 4y' + 4y\} = \mathcal{L}\{t^3e^{2t}\}$$

$$s^2Y(s) - s\underbrace{y(0)}_{0} - \underbrace{y'(0)}_{0} - 4(sY(s) - \underbrace{y(0)}_{0}) + 4Y(s) = \mathcal{L}\{t^3\}\big|_{s\to s-2}$$

$$(s^2 - 4s + 4)Y(s) = \frac{3!}{(s-2)^4} \Rightarrow Y(s) = \frac{6}{(s-2)^6}$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}\left\{\frac{6}{(s-2)^6}\right\} = 6\mathcal{L}^{-1}\left\{\frac{1}{s^6}\Big|_{s\to s-2}\right\} = 6 \cdot e^{2t}\frac{1}{5!}t^5$$

$$\Rightarrow y(t) = \frac{1}{20}t^5e^{2t}$$

Undetermined Coefficients

(use the y_c to the right)

$$y_p = (At^5 + Bt^4 + Ct^3 + Dt^2)e^{2t}$$

(Initial guess $y_p = (At^3 + Bt^2 + Ct + D)e^{2t}$ has to be multiplied by t^2 to eliminate duplication of y_c).

$$\Rightarrow y_p' = (2At^5 + (5A + 2B)t^4 + (4B + 2C)t^3 + (3C + 2D)t^2 + 2Dt)e^{2t}$$

$$\Rightarrow y_p'' = \big(4At^5 + (20A + 4B)t^4 + (20A + 16B + 4C)t^3 + (12B + 12C + 4D)t^2 + (6C + 8D)t + 2D\big)e^{2t}$$

Put back into original eqn. and match coefficients:

t^5e^{2t}: $4A - 8A + 4A = 0$ ✓

t^4e^{2t}: $20A + 4B - 20A - 8B + 4B = 0$ ✓

t^3e^{2t}: $20A + 16B + 4C - 16B - 8C + 4C = 1$

$\Rightarrow 20A = 1 \Rightarrow A = 1/20$

t^2e^{2t}: $12B + 12C + 4D - 12C - 8D + 4D = 0$

$\Rightarrow 12B = 0 \Rightarrow B = 0$

te^{2t}: $6C + 8D - 8D = 0 \Rightarrow 6C = 0 \Rightarrow C = 0$

e^{2t}: $2D = 0 \Rightarrow D = 0$

$$\Rightarrow y_p = \frac{1}{20}t^5e^{2t}$$

(Go back and add y_c, find $c_1 = c_2 = 0$).

Variation of Parameters

① y_c: $m^2 - 4m + 4 = 0 \Rightarrow (m-2)^2 = 0$

$$y_c = c_1e^{2t} + c_2te^{2t}$$

② y_p:

$$W = \begin{vmatrix} e^{2t} & te^{2t} \\ 2e^{2t} & (2t+1)e^{2t} \end{vmatrix} = e^{4t}$$

$$W_1 = \begin{vmatrix} 0 & te^{2t} \\ t^3e^{2t} & (2t+1)e^{2t} \end{vmatrix} = -t^4e^{4t}$$

$$\Rightarrow u_1' = -t^4 \Rightarrow u_1 = -\frac{1}{5}t^5$$

$$W_2 = \begin{vmatrix} e^{2t} & 0 \\ 2e^{2t} & t^3e^{2t} \end{vmatrix} = t^3e^{4t}$$

$$\Rightarrow u_2' = t^3 \Rightarrow u_2 = \frac{1}{4}t^4$$

$$\Rightarrow y_p = -\frac{1}{5}t^5e^{2t} + \frac{1}{4}t^5e^{2t} = \frac{1}{20}t^5e^{2t}$$

③ $y := c_1e^{2t} + c_2te^{2t} + \frac{1}{20}t^5e^{2t}$

④ Find c_1, c_2: $y(0) = 0 \Rightarrow c_1 = 0$

$y'(0) = 0 \Rightarrow c_2 = 0$

$$\Rightarrow y(t) = \frac{1}{20}t^5e^{2t}$$

What is the conclusion?

It is often easier to solve the type of problem: Linear ODE w/ constant coeff. IVP w/ conditions at 0 using the Laplace transform. It gets straight to the point and just becomes an $\mathcal{L}^{-1}$ problem. Keyword here is IVP – if you want the general solution, you can use Laplace to maybe find y_p, but you must still find y_c.

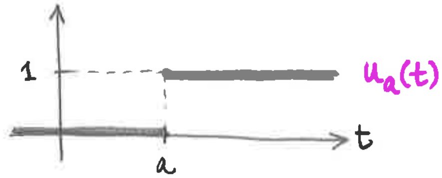
Laplace Transform & the Unit Step Function

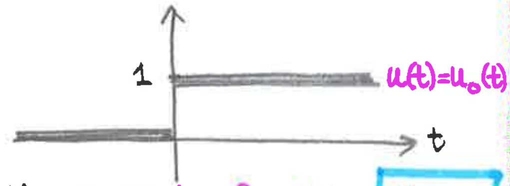
Def.: The Unit Step Function (aka the **Heaviside function**):

$$u(t) = \begin{cases} 0, & t<0 \\ 1, & t\geq 0 \end{cases}$$

- models situations where a signal can either be "on" or "off".
- Translations of $u(t)$ allows one to turn signals off at times other than 0:

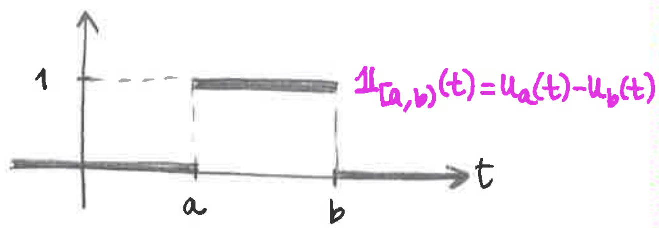
$$u_a(t) = \begin{cases} 0, & t<a \\ 1, & t\geq a \end{cases}$$



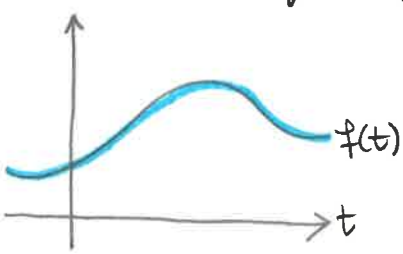


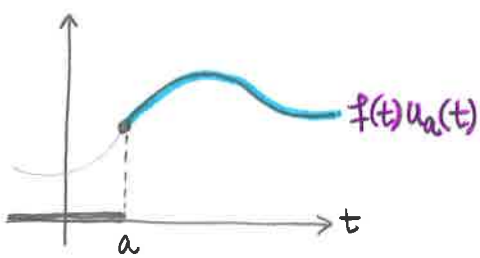
- We can turn a signal "on" at $t=a$ and "off" at $t=b$, using the **indicator function** $\mathbb{1}_{[a,b)}$ on $[a,b)$: "on" on $[a,b)$, "off" elsewhere:

$$\mathbb{1}_{[a,b)}(t) = u_a(t) - u_b(t) = \begin{cases} 0, & t<a \\ 1, & a\leq t<b \\ 0, & t>b \end{cases}$$

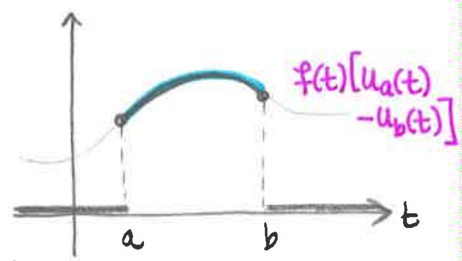


- Multiply a function $f(t)$ by $u_a(t)$ or $\mathbb{1}_{[a,b)}(t)$ to "turn off" portions of its graph: (also makes it easy to express piecewise-defined functions).





("turns off" the graph of f before t=a)



("turns off" the graph of f everywhere outside $a\leq t<b$).

- $$\mathcal{L}\{u_a(t)\} = \frac{e^{-as}}{s}\ ;\ s>0$$

$$\mathcal{L}\{u_a(t)\} = \int_a^\infty e^{-st}\,dt = -\frac{1}{s}e^{-st}\Big|_{t=a}^{\infty} = \frac{e^{-as}}{s}\ ;\ s>0.$$

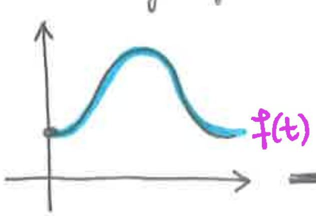
The Second Translation Theorem:

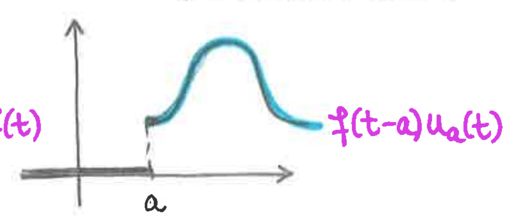
$(a>0)$ $$\mathcal{L}\{f(t-a)u_a(t)\} = e^{-as}F(s)$$

Inverse Form:

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)u_a(t)$$ $(a>0)$

Remark: What does $f(t-a)u_a(t)$ do? Say we have a function $f(t)$ defined on $[0,\infty)$. Translating by $a>0$, i.e. $f(t-a)$ "shifts" the graph of f to the right by a units, and is now a function defined on $[a,\infty)$. Multiplying by $u_a(t)$ does not change this, but only defines the new function to be 0 on $[0,a)$:





Proof: $\mathcal{L}\{f(t-a)u_a(t)\} = \int_0^\infty e^{-st} f(t-a)\underbrace{u_a(t)}_{\substack{0 \text{ before } t=a \\ 1 \text{ after}}}dt = \underbrace{\int_a^\infty e^{-st} f(t-a)\,dt}_{\text{change of variable: } t'=t-a}$

$\Rightarrow dt' = dt$
$\Rightarrow t=a \Rightarrow t'=0$
$\Rightarrow t\to\infty \Rightarrow t'\to\infty$

$$= \int_0^\infty e^{-s(t'+a)} f(t')\,dt' = e^{-as}\underbrace{\int_0^\infty e^{-st'} f(t')\,dt'}_{\mathcal{L}\{f(t)\}=F(s).}$$

Ex: $f(t) = \begin{cases} (t-2)^3, & t \geq 2 \\ 0, & 0 \leq t < 2 \end{cases}$

Express in terms of step functions:

$f(t) = (t-2)^3 u_2(t)$

$\Rightarrow \mathcal{L}\{f(t)\} = \mathcal{L}\{(t-2)^3 u_2(t)\} = e^{-2s}\mathcal{L}\{t^3\} = \frac{6}{s^4}e^{-2s}$

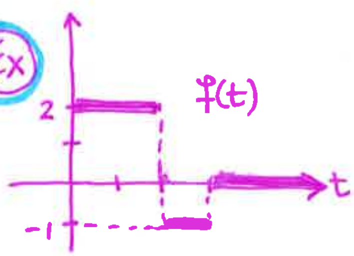
Ex: $f(t) = \begin{cases} (t-2)^3, & t \geq 3 \\ 0, & 0 \leq t < 3 \end{cases}$

$f(t) = (t-2)^3 u_3(t)$

The Translation Thm. does not immediately apply – the $u_3(t)$ suggests it needs things in terms of $\underline{\underline{(t-3)}}$:

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{(t-2)^3 u_3(t)\} = \mathcal{L}\{[(t-3)+1]^3 u_3(t)\} = e^{-3s}\mathcal{L}\{(t+1)^3\}$$
$$= e^{-3s}\mathcal{L}\{t^3+3t^2+3t+1\} = e^{-3s}\left(\frac{6}{s^4}+\frac{6}{s^3}+\frac{3}{s^2}+\frac{1}{s}\right).$$

Ex

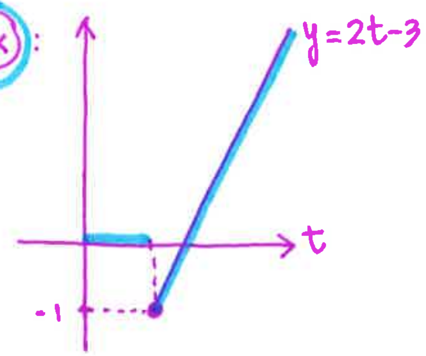


Express in terms of step functions:

$$f(t) = 2\big(\underbrace{u_0(t)-u_2(t)}_{\mathbb{1}_{[0,2)}}\big) - 1\big(\underbrace{u_2(t)-u_3(t)}_{\mathbb{1}_{[2,3)}}\big)$$
$$= 2u_0(t) - 3u_2(t) + u_3(t)$$
$$\Rightarrow \mathcal{L}\{f(t)\} = 2\mathcal{L}\{u_0(t)\} - 3\mathcal{L}\{u_2(t)\} + \mathcal{L}\{u_3(t)\}$$
$$= 2\cdot\frac{1}{s} - 3\cdot\frac{e^{-2s}}{s} + \frac{e^{-3s}}{s}.$$

Ex: $\mathcal{L}\{\sin(t)u_{2\pi}(t)\} = \mathcal{L}\{\sin(t-2\pi)u_{2\pi}(t)\} = e^{-2\pi s}\mathcal{L}\{\sin t\} = \frac{e^{-2\pi s}}{s^2+1}.$
(sin is periodic w/ period 2π).

Ex:



$f(t) =$ the line $y=2t-3$ "turned off" on $[0,1)$:

$$f(t) = (2t-3)u_1(t)$$
$$\Rightarrow \mathcal{L}\{f(t)\} = \mathcal{L}\{(2t-3)u_1(t)\} = \mathcal{L}\{(2(t-1)-1)u_1(t)\}$$
$$= e^{-s}\mathcal{L}\{2t-1\} = e^{-s}(2\mathcal{L}\{t\} - \mathcal{L}\{1\})$$
$$= e^{-s}\left(\frac{4}{s^2}-\frac{1}{s}\right).$$

Ex: $\mathcal{L}^{-1}\left\{\frac{e^{-\pi s/2}}{s^2+9}\right\} = \mathcal{L}^{-1}\left\{e^{-\frac{\pi}{2}s}\frac{1}{s^2+9}\right\}$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\}\Big|_{t\to t-\pi/2} u_{\pi/2}(t) = \frac{1}{3}\sin(3t)\Big|_{t\to t-\pi/2} u_{\pi/2}(t)$$

$$= \frac{1}{3}\sin(3t - 3\pi/2)\, u_{\pi/2}(t)$$

$$= \frac{1}{3}\cos(3t)\, u_{\pi/2}(t).$$

$\sin(3t-3\pi/2) =$
$= \sin(3t - 3\pi/2 + 2\pi)$
$= \sin(3t+\pi/2) = \cos(3t)$

Laplace Transform of Periodic Functions

Periodic function f with period T: $f(t) = f(t+T),\ \forall t>0.$

Laplace transforms of periodic functions can be obtained by integration over 1 period:

Suppose $f(t)$ is piecewise continuous on $t\in[0,\infty)$, of exponential order, and periodic with period T. Then

$$\boxed{\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\,dt} = \boxed{\frac{1}{1-e^{-sT}}\mathcal{L}\{f(t)(1-u_T(t))\}}$$

Proof.: $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = \int_0^T e^{-st} f(t)\,dt + \underbrace{\int_T^\infty e^{-st} f(t)\,dt}$

$t = u+T$
$dt = du$

$$= \int_0^\infty e^{-s(u+T)} f(u+T)\,du$$

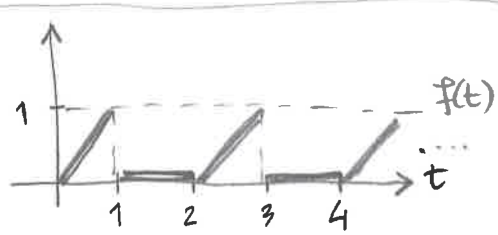
$$= e^{-sT}\int_0^\infty e^{-su} f(u)\,du = e^{-sT}\mathcal{L}\{f(t)\}$$

$$\Rightarrow (1-e^{-sT})\mathcal{L}\{f(t)\} = \int_0^T e^{-st} f(t)\,dt$$

$$= \int_0^\infty e^{-st} f(t)(1-u_T(t))\,dt$$

$$= \mathcal{L}\{f(t)(1-u_T(t))\}.$$

Ex:



Period: $T=2$

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-2s}}\int_0^2 e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2s}}\int_0^1 e^{-st}\, t\,dt$$

$$= \frac{1}{1-e^{-2s}}\mathcal{L}\{t(1-u_1(t))\}$$

$$= \frac{1-e^{-s}-se^{-s}}{(1-e^{-2s})s^2}$$

$$\mathcal{L}\{t(1-u_1(t))\} = \mathcal{L}\{t\} - \mathcal{L}\{t u_1(t)\}$$

$$= \frac{1}{s^2} - \mathcal{L}\{(t-1)u_1(t)\} - \mathcal{L}\{u_1(t)\}$$

$$= \frac{1}{s^2} - e^{-s}\frac{1}{s^2} - e^{-s}\frac{1}{s} = \frac{1-e^{-s}-se^{-s}}{s^2}$$