HIGHER ORDER LINEAR ODEs

n^{th} order linear ODE: $a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \dots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$

IVP: The dependent variable y and its derivatives are specified at the same value x_0 of x (indp. var.):

$y(x_0) = b_0$; $y'(x_0) = b_1$; … ; $y^{(n-1)}(x_0) = b_{n-1}$

for some numbers $b_0, b_1, \dots, b_{n-1}$

BVP: The dependent variable y and its derivatives are specified at different values of x (indp. var.)

Ex: For a 2^{nd} order linear eqn:

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x)$$

one can have the following possibilities for the boundary value conditions:

- $y(a) = b_0$; $y(b) = b_1$
- $y'(a) = b_0$; $y(b) = b_1$
- $y(a) = b_0$; $y'(b) = b_1$
- $y'(a) = b_0$; $y'(b) = b_1$

Homogeneous if $g = 0$

(different meaning of the word "homogeneous")

Non-Homogeneous if $g \neq 0$

Ex: $y''' + 2y'' - e^x y = x^2$ (non-homogeneous)

$y^{(4)} - xe^x y'' + y = 0$ (homogeneous)

- To solve a non-homogeneous linear ODE, we must first solve the homogeneous one.

Existence & Uniqueness for IVPs:

Let an n^{th} order linear ODE: $a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_0 y = g$; $y(x_0) = b_0, \dots, y^{(n-1)}(x_0) = b_{n-1}$

Suppose that an interval $I \subset \mathbb{R}$ satisfies:

- $a_n(x), a_{n-1}(x), \dots, a_0(x)$ and $g(x)$ are continuous on I.
- $a_n(x) \neq 0$ for all $x \in I$.

Then, for every $x_0 \in I$, there exists a unique solution to the IVP.

Ex: $y = 3e^{2x} + e^{-2x} - 3x$ is a solution to: $y'' - 4y = 12x$; $y(0) = 4$; $y'(0) = 1$

By the Theorem above, it is the unique solution. $a_2(x) = 1$ continuous, never 0
$a_1(x) = 0$ continuous
$a_0(x) = -4$ continuous
$g(x) = 12x$ continuous.

Ex: $y = cx^2 - x$ is a solution to $x^2 y'' - 2xy' + 2y = 0$; $y(0) = 0$; $y'(0) = -1$ for any value of c
(an IVP with ∞-many solutions).
(the assumptions of the Thm. are not met: $a_2(x) = x^2$; $a_2(0) = 0$.)

- Even when the assumptions of the Thm. above are satisfied, a BVP can have
  - * several solutions (∞-many)
  - * one solution
  - * no solutions.

Ex: $y'' + 16y = 0$

$y = c_1 \cos(4x) + c_2 \sin(4x)$ – general solution –

Impose different boundary conditions:

- BVP 1: $y(0) = 0$; $y(\pi/2) = 0$ $\Rightarrow$ ∞-many solutions ($y = c_2 \sin(4x)$ is a sol. for all c_2)
- BVP 2: $y(0) = 0$; $y(\pi/8) = 0$ $\Rightarrow$ one solution ($y = 0$)
- BVP 3: $y(0) = 0$; $y(\pi/2) = 1$ $\Rightarrow$ no solutions

Linear Independence

Def.: A set of n functions $f_1(x), \dots, f_n(x)$ are called <u>linearly dependent</u> on some interval $I \subset \mathbb{R}$ if there exist constants $c_1, c_2, \dots, c_n$ (not all zero) such that:

$$\boxed{c_1 f_1(x) + c_2 f_2(x) + \dots + c_n f_n(x) = 0 \ , \ \forall x \in I}$$

Otherwise, they are called <u>linearly independent</u>.

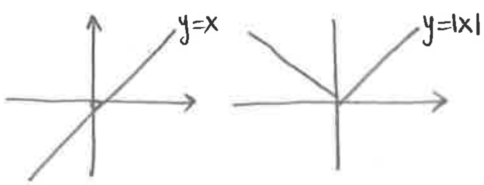
Ex: $f_1(x) = x^2$; $f_2(x) = 3x^2$

$-3f_1(x) + f_2(x) = -3x^2 + 3x^2 = 0, \forall x \in \mathbb{R}$

$\Rightarrow$ linearly dependent on $\mathbb{R}$

Two functions $f_1(x), f_2(x)$ are lin. dep. if and only if they are constant multiples of one another.

Ex: $f_1(x) = x$; $f_2(x) = |x|$



$\rightarrow$ lin. indp. on $\mathbb{R}$ (cannot be expressed as constant multiples of each other on $\mathbb{R}$)

$\rightarrow$ lin. dep. on $(0, \infty)$ $\quad x = 1 \cdot |x|$ on $(0, \infty)$

$\rightarrow$ lin. dep. on $(-\infty, 0)$ $\quad x = -1 \cdot |x|$ on $(-\infty, 0)$.

Ex: $f_1(x) = \sqrt{x} + 3$

$f_2(x) = \sqrt{x} + 3x$

$f_3(x) = x - 1$

$f_4(x) = x^2$

$(f_2 - f_1)(x) = 3x - 3$

$= 3(x-1) = 3f_3(x)$

$\Rightarrow f_1(x) - f_2(x) + 3f_3(x) + 0 \cdot f_4(x) = 0 \Rightarrow$ lin. dep. on $(0, \infty)$

the constants can't all be 0 but it's allowed for some to be 0.

Wronskian

Suppose $f_1, f_2, \dots, f_n$ have at least $(n-1)$ derivatives. Define their Wronskian as:

$$W(f_1, \dots, f_n)(x) = \begin{vmatrix} f_1(x) & f_2(x) & \dots & f_n(x) \\ f_1'(x) & f_2'(x) & \dots & f_n'(x) \\ \vdots & & & \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \dots & f_n^{(n-1)}(x) \end{vmatrix}$$

- If $f_1, f_2, \dots, f_n$ are <u>linearly dependent</u> on I, then $W(f_1, \dots, f_n)(x) = 0, \ \forall x \in I$.

$\Rightarrow$ If $W(f_1, \dots, f_n)(x) \neq 0$ for at least one value of x on I, then $f_1, \dots, f_n$ are <u>linearly independent</u>.

Proof ($n=2$): Suppose $W(f_1, f_2)(x) \neq 0$ for at least one $x \in I$, and also that f_1, f_2 are lin. dep. $\Rightarrow \exists$ constants (not both 0) c_1, c_2 s.t. $c_1 f_1 + c_2 f_2 = 0$ on I.

Assume $c_1 \neq 0$.

$c_1 f_1 + c_2 f_2 = 0 \ | \cdot f_2'$ $\qquad c_1 f_1 f_2' + c_2 f_2 f_2' = 0$

$c_1 f_1' + c_2 f_2' = 0 \ | \cdot f_2$ $\qquad c_1 f_1' f_2 + c_2 f_2 f_2' = 0$

$\ominus \qquad c_1 (f_1 f_2' - f_1' f_2) = 0$

$\Rightarrow f_1 f_2' - f_1' f_2 = 0$

$W(f_1, f_2) = 0$

$\Rightarrow$ contradiction.

Higher Order Linear ODEs

(A) n^{th} Order Linear Homogeneous Equations

$$\boxed{a_n(x)y^{(n)}(x)+a_{n-1}(x)y^{(n-1)}(x)+\ldots+a_1(x)y'(x)+a_0(x)y(x)=0} \quad (*)$$

(assume throughout that we are working on an interval I where the Existence & Uniqueness Thm. hold)

Superposition Principle:

If $y_1, y_2, \ldots, y_K$ are solutions to $(*)$, where K is some positive integer, then any linear combination:

$$y = C_1y_1 + C_2y_2 + \ldots + C_Ky_K$$

is also a solution to $(*)$.

To see this, look at the case $K=2$, so suppose y_1 and y_2 are solutions to $(*)$:

$$\begin{cases} a_n y_1^{(n)} + a_{n-1}y_1^{(n-1)} + \ldots + a_1 y_1' + a_0 y_1 = 0 & \Big| \cdot C_1 \quad \text{(some constant)} \\ a_n y_2^{(n)} + a_{n-1}y_2^{(n-1)} + \ldots + a_1 y_2' + a_0 y_2 = 0 & \Big| \cdot C_2 \quad \text{(some constant)} \end{cases}$$

$$\Rightarrow \begin{cases} a_n(C_1y_1)^{(n)} + a_{n-1}(C_1y_1)^{(n-1)} + \ldots + a_1(C_1y_1)' + a_0(C_1y_1) = 0 \\ a_n(C_2y_2)^{(n)} + a_{n-1}(C_2y_2)^{(n-1)} + \ldots + a_1(C_2y_2)' + a_0(C_2y_2) = 0 \end{cases}$$

$$\oplus \quad a_n(C_1y_1+C_2y_2)^{(n)} + \ldots + a_1(C_1y_1+C_2y_2)' + a_0(C_1y_1+C_2y_2) = 0$$

$\Rightarrow C_1y_1 + C_2y_2$ is also a solution!

Take a moment to remark why this can only work for homogeneous ODEs!

Some Useful Consequences:

(1) If y_1 is a solution to $(*)$, then any constant multiple $y = Cy_1$ is also a solution. (Take $K=1$ above)

(2) The trivial solution $y=0$ satisfies any homogeneous linear ODE. (Take $C_1 = \ldots = C_K = 0$ or just verify).

Fundamental Sets

Def.: A fundamental set of solutions to an n^{th} order linear homogeneous ODE (*) on some interval I is a set:

$$\{y_1, y_2, \dots, y_n\}$$

of n solutions that are linearly independent on I.

- Two very important facts about fundamental sets:
  1). If we have a fundamental set, we have all the solutions.
  2). An "if and only if" relationship to the Wronskian.

1 Let $\{y_1, y_2, \dots, y_n\}$ be a fundamental set of solutions to (*) on some interval I. Then ANY solution to (*) on I is a linear combination of $y_1, \dots, y_n$.

For this reason, if $\{y_1, \dots, y_n\}$ is a fundamental set, then

$$\boxed{y = c_1 y_1 + c_2 y_2 + \dots + c_n y_n}$$

is called the general solution to (*) on I.

This is actually a simple consequence of the Existence & Uniqueness Theorem and a result from linear algebra, which we recall below:

Linear Algebra Flashback:

Recall that a system of n linear algebraic equations:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n \end{cases}$$

can be expressed in matrix form as $A\vec{x} = \vec{b}$, where A is the $n \times n$ matrix of coefficients, $\vec{x}$ is the vector of unknowns.

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} ; \quad \vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} ; \quad \vec{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}.$$

Possible behaviors of such a system, in terms of the determinant $\det(A)$:

- $\det(A) \neq 0 \Rightarrow$ unique solution
- $\det(A) = 0 \Rightarrow$ either no solution or infinitely many solutions.

[Important special case: if the system is homogeneous, i.e. $\vec{b} = \vec{0}$ and $\det(A) = 0$, then there are infinitely many solutions]

Back to our homogeneous ODEs, look at the case $\boxed{n=2}$.

Equation:

$$\boxed{a_2(x)y''(x) + a_1(x)y'(x) + a_0(x)y(x) = 0} \qquad (*)$$

(assume a_2, a_1, a_0 are continuous on I and $a_2(x) \neq 0, \forall x \in I \rightsquigarrow \exists!$)

Suppose:
- $Y(x)$ = a solution to $(*)$
- $\{y_1(x), y_2(x)\}$ = a <u>fundamental set</u> for $(*)$ (i.e. <u>lin. indp</u>, solutions).

Why should Y be a linear combination of y_1 and y_2?

- Let $x_0 \in I$ be a point where the Wronskian $W(y_1, y_2)(x_0) \neq 0$
(we know such a point must exist, since y_1, y_2 are <u>linearly independent</u>).
- Let $b_1 = Y(x_0)$, $b_2 = Y(x_0)$ and look at the system:

$$\begin{cases} y_1(x_0)c_1 + y_2(x_0)c_2 = b_1 \\ y_1'(x_0)c_1 + y_2'(x_0)c_2 = b_2 \end{cases}$$

(c_1, c_2 = the unknowns
$y_{1,2}(x_0)$, $y_{1,2}'(x_0)$ = the coefficient)

The determinant of coefficients:

$$\begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix} = W(y_1, y_2)(x_0) \neq 0$$

$\Rightarrow$ the system has a unique solution c_1, c_2.

- Then look at the function $Z(x) = c_1 y_1(x) + c_2 y_2(x)$:
  - $*$ Z is a solution to $(*)$ $\rightsquigarrow$ <u>Superposition</u>
  - $*$ $Z(x_0) = b_1$ and $Z(x_0) = b_2$

$\Rightarrow$ Y and Z are <u>both</u> solutions on I to the <u>IVP</u> $\begin{cases} a_2 y'' + a_1 y' + a_0 y = 0 \\ y(x_0) = b_1; \ y'(x_0) = b_2 \end{cases}$

$\Rightarrow$ $\boxed{Y = Z}$ by the Existence & Uniqueness Theorem

$\Rightarrow$ $Y = c_1 y_1 + c_2 y_2$. ∎

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(2) A set $\{y_1, \dots, y_n\}$ of solutions to $(*)$ is a fundamental set (i.e. they are linearly independent) on an interval I <u>if and only if</u> the Wronskian:

$$\boxed{W(y_1, y_2, \dots, y_n)(x) \neq 0, \ \forall x \in I.}$$

$*$ <u>What is special here?</u> One implication is easy: we know from last time that if $W(f_1, \dots, f_n)(x) \neq 0$ for <u>at least one</u> $x \in I$, then $\{f_1, \dots, f_n\}$ are lin. indep. (so obviously also if $W \neq 0$ for <u>all</u> x).

$*$ Special: If $\{y_1, \dots, y_n\}$ = fundamental set $\Rightarrow$ $W(y_1, \dots, y_n)(x) \neq 0$ for <u>all</u> $x \in I$.

(B) n^{th} Order Linear Non-Homogeneous Equations

$$\boxed{a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \ldots + a_1(x)y'(x) + a_0(x)y(x) = g(x)} \quad (**)$$

- **Particular Solution**: A particular solution to (**) is any solution y_p that is free of any arbitrary constants.

- **Complementary Solution**: Every non-homogeneous linear equation (**) has an associated homogeneous linear equation (*)

$$\boxed{a_n(x)y^{(n)}(x) + \ldots + a_1(x)y'(x) + a_0(x)y(x) = 0} \quad (*)$$

A general solution $y_c = c_1y_1 + \ldots + c_ny_n$ to the homogeneous equation (*) is called a complementary solution (or complementary function).

ANY solution of the non-homogeneous equation (**) is of the form

$$\boxed{\begin{aligned} y &= y_c + y_p \\ &= (c_1y_1 + \ldots + c_ny_n) + y_p \end{aligned}}$$

where y_c is the complementary solution and y_p is any particular solution.

Ex: Consider the equation $\boxed{y'' - 9y = 27}$

Given that the general solution to $\boxed{y'' - 9y = 0}$ is $\boxed{y_c = c_1e^{3x} + c_2e^{-3x}}$ (check)

you can easily see that $\boxed{y_p = -3}$ (constant function) is a particular solution.

Then $\boxed{y = c_1e^{3x} + c_2e^{-3x} - 3}$ is the general solution to this non-homogeneous equation.

Proof.: Let Y be any solution to (**). Since y_p is also a solution:

$$\begin{aligned} &a_nY^{(n)} + a_{n-1}Y^{(n-1)} + \ldots + a_1Y' + a_0Y = g \\ &a_ny_p^{(n)} + a_{n-1}y_p^{(n-1)} + \ldots + a_1y_p' + a_0y_p = g \\ \hline \ominus\ &a_n(Y-y_p)^{(n)} + \ldots + a_1(Y-y_p)' + a_0(Y-y_p) = \underline{\underline{0}} \end{aligned}$$

$\Rightarrow (Y - y_p)$ = solution to the homogeneous equation

$\Rightarrow (Y - y_p)$ = linear combination of fundamental set

$\Rightarrow Y - y_p = y_c$.

Homogeneous Linear ODEs with Constant Coefficients

$$\boxed{\begin{gathered} a_n y^{(n)} + \ldots + a_1 y' + a_0 y = 0 \\ a_n, a_{n-1}, \ldots, a_1, a_0 \in \mathbb{R} \end{gathered}}$$

$\boxed{n=2}$: $ay'' + by' + c = 0$

Characteristic Equation: $\boxed{am^2 + bm + c = 0}$

- 2 distinct real roots m_1, m_2 : $y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$
- 1 repeated real root m_1 : $y = c_1 e^{m_1 x} + c_2 \underline{\underline{x}} e^{m_1 x}$
- 2 complex roots $m_{1,2} = \alpha \pm i\beta$: $y = e^{\alpha x}\left(c_1 \cos(\beta x) + c_2 \sin(\beta x)\right)$

$\boxed{\text{General } n}$: $a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = 0$

Characteristic Equation: $\boxed{a_n m^n + a_{n-1} m^{n-1} + \ldots + a_1 m + a_0 = 0}$

- For any real root m_1 of multiplicity $\underline{\underline{1}}$, the general solution must contain $c_1 e^{m_1 x}$
- For any real root m_1 of multiplicity $\underline{\underline{k}}$, the general solution must contain $c_1 e^{m_1 x} + c_2 x e^{m_1 x} + \ldots + c_k x^{k-1} e^{m_1 x}$
- For any pair of complex roots $\alpha \pm i\beta$, the general solution must contain $e^{\alpha x}\left(c_1 \cos(\beta x) + c_2 \sin(\beta x)\right)$
- If a complex root $\alpha + i\beta$ has multiplicity $\underline{\underline{k}}$, then so does its conjugate $\alpha - i\beta$, and the general solution must contain <u>linear combinations</u> of
$$\left\{\begin{array}{c} e^{\alpha x}\cos(\beta x),\ e^{\alpha x}\sin(\beta x) \\ x e^{\alpha x}\cos(\beta x),\ x e^{\alpha x}\sin(\beta x) \\ \vdots \\ x^{k-1} e^{\alpha x}\cos(\beta x),\ x^{k-1} e^{\alpha x}\sin(\beta x) \end{array}\right\}$$

Examples :

- $y'' - 2y' + y = 0$

$m^2 - 2m + 1 = 0$

$(m-1)^2 = 0 \Rightarrow m = 1$ (repeated root) $\Rightarrow \boxed{y = c_1 e^x + c_2 x e^x}$

- $y'' + y' + y = 0$

$m^2 + m + 1 = 0$

$\Delta = 1 - 4 = -3 \Rightarrow m = \dfrac{-1 \pm i\sqrt{3}}{2} = \dfrac{-1}{2} \pm i\dfrac{\sqrt{3}}{2} \Rightarrow$

$$\boxed{y = e^{-x/2}\left(c_1 \cos\left(\tfrac{\sqrt{3}}{2}x\right) + c_2 \sin\left(\tfrac{\sqrt{3}}{2}x\right)\right)}$$

- $y'' + y' - 2y = 0$

$m^2 + m - 2 = 0$

$(m+2)(m-1) = 0$

$m_1 = -2;\ m_2 = 1$

$$\boxed{y = c_1 e^{-2x} + c_2 e^x}$$

Characteristic Equation :	General Solution :
$(m-2)(m^2+m+1)$ 2 $\quad -1/2 \pm \sqrt{3}/2\, i$	$c_1 e^{2x} + e^{-x/2}\left(c_2 \sin\left(\tfrac{\sqrt{3}}{2}x\right) + c_3 \cos\left(\tfrac{\sqrt{3}}{2}x\right)\right)$
$(m-2)^3 (m^2+16)$ 2×3 $\quad \pm 4i$	$c_1 e^{2x} + c_2 x e^{2x} + c_3 x^2 e^{2x}$ $+ c_4 \sin(4x) + c_5 \cos(4x)$
$(m+3)(m-2)^2(m-5)^3$ -3 $\quad 2 \times 2$ $\quad 5 \times 3$	$c_1 e^{-3x} + c_2 e^{2x} + c_3 x e^{2x}$ $+ c_4 e^{5x} + c_5 x e^{5x} + c_6 x^2 e^{5x}$
$(m^2+m+1)^2$ $-1/2 \pm \sqrt{3}/2\, i$ $\times 2$	$e^{-x/2}\left(c_1 \sin(\sqrt{3}/2\, x) + c_2 \cos(\sqrt{3}/2\, x)\right)$ $+ e^{-x/2} x \left(c_3 \sin(\sqrt{3}/2\, x) + c_4 \cos(\sqrt{3}/2\, x)\right).$

Non-homogeneous Linear ODEs with Constant Coefficients

– Method of Undetermined Coefficients –

Method applies to equations of the form:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = g(x)$$

where:

- $a_n, a_{n-1}, \ldots, a_1, a_0$ are constants
- $g(x)$ is :
  - a polynomial (includes constant functions)
  - exponential $e^{\alpha x}$
  - $\sin(\beta x)$ or $\cos(\beta x)$
  - sums and products of these

General Idea : This method is for finding a particular solution y_p.

- Make an assumption about the form of y_p (some type of function with unknown coefficients), then find the coefficients.
- General rule for the assumption : linear combination of all the linearly independent types of functions that arise from repeated differentiations of $g(x)$.
- Polynomials, exponentials, and sin & cos work because any sum or product of these gives a sum or product of these same types of functions through repeated differentiation.

Superposition Principle for Non-Homogeneous Linear ODEs :

Suppose that for every $j \in \{1, 2, .., k\}$, y_{p_j} is a particular solution for

$$a_n(x) y^{(n)} + \ldots + a_1(x) y' + a_0(x) y = g_j(x).$$

Then $y_{p_1} + \ldots + y_{p_k}$ is a particular solution for

$$a_n(x) y^{(n)} + \ldots + a_1(x) y' + a_0(x) y = g_1(x) + \ldots + g_k(x).$$

Case 1: No function in the assumed y_p duplicates any part of y_c (i.e. no function in the assumed y_p is a solution to the associated homogeneous equation).

$g(x)$	y_p
Polynomial $p(x)$ of degree $k \geq 0$	Polynomial of same degree k
$g(x) = x+1$	$y_p = Ax+B$
$g(x) = x^3+x-4$	$y_p = Ax^3+Bx^2+Cx+D$
$g(x)=2$	$y_p = A$
Exponential $e^{\alpha x}$	$Ae^{\alpha x}$
$g(x) = e^{9x}$	$y_p = Ae^{9x}$
$\sin(\beta x)$ or $\cos(\beta x)$	$A\sin(\beta x) + B\cos(\beta x)$
$g(x) = \sin(2x)$	$y_p = A\sin(2x)+B\cos(2x)$
$g(x) = \cos(7x)$	$y_p = A\sin(7x)+B\cos(7x)$
$p(x)e^{\alpha x}$	(same degree polynomial) $\cdot\, e^{\alpha x}$
$g(x) = (x+1)e^{3x}$	$y_p = (Ax+B)e^{3x}$
$p(x)\sin(\beta x)$ or $p(x)\cos(\beta x)$	(Same deg. poly) $\cdot \sin(\beta x)$ + (Same deg. poly) $\cdot \cos(\beta x)$
$g(x) = (x-1)\sin(2x)$	$y_p = (Ax+B)\sin(2x) + (Cx+D)\cos(2x)$
$g(x) = x^2\cos(3x)$	$y_p = (Ax^2+Bx+C)\sin(3x) + (Dx^2+Ex+F)\cos(3x)$
$e^{\alpha x}\sin(\beta x)$ or $e^{\alpha x}\cos(\beta x)$	$Ae^{\alpha x}\sin(\beta x) + Be^{\alpha x}\cos(\beta x)$
$g(x) = e^{2x}\sin(7x)$	$y_p = Ae^{2x}\sin(7x) + Be^{2x}\cos(7x)$
$p(x)e^{\alpha x}\sin(\beta x)$ or $p(x)e^{\alpha x}\cos(\beta x)$	(Same deg. poly.) $e^{\alpha x}\sin(\beta x)$ + (Same deg. poly.) $e^{\alpha x}\cos(\beta x)$
$g(x) = (x+1)e^{2x}\cos(3x)$	$y_p = (Ax+B)e^{2x}\sin(3x) + (Cx+D)e^{2x}\cos(3x)$
SUMS OF FUNCTIONS OF THE TYPES ABOVE	SUM OF INDIVIDUAL GUESSES FOR y_p (SUPERPOSITION)
$g(x) = x^2 + e^{7x}\cos(\pi x)$	$y_p = Ax^2+Bx+C + De^{7x}\sin(\pi x) + Ee^{7x}\cos(\pi x)$
$g(x) = x\sin(2x) + e^{3x} + x^3$	$y_p = (Ax+B)\sin(2x) + (Cx+D)\cos(2x) + Ee^{3x} + Fx^3+Gx^2+Hx+I.$

Example 1: $y'' - 2y' - 3y = e^{2x} + 3x^2 + 4x - 5 + 5\cos(2x)$

Complementary Solution: $y'' - 2y' - 3y = 0$

$y_c = c_1 e^{-x} + c_2 e^{3x}$ $\quad m^2 - 2m - 3 = 0 \Rightarrow (m-3)(m+1) = 0 \Rightarrow m \in \{-1, 3\}$

Particular Solution: $y_p = Ae^{2x} + Bx^2 + Cx + D + E\cos(2x) + F\sin(2x)$

$$y_p' = 2Ae^{2x} + 2Bx + C - 2E\sin(2x) + 2F\cos(2x)$$

$$y_p'' = 4Ae^{2x} + 2B - 4E\cos(2x) - 4F\sin(2x)$$

$$\Rightarrow y_p'' - 2y_p' - 3y_p = e^{2x}(\underbrace{4A - 4A - 3A}_{1}) + x^2(\underbrace{-3B}_{3}) + x(\underbrace{-4B - 3C}_{4}) + (\underbrace{2B - 2C - 3D}_{-5})$$

$$+ \cos(2x)(\underbrace{-4E - 4F - 3E}_{5}) + \sin(2x)(\underbrace{-4F + 4E - 3F}_{0})$$

$$\begin{cases} -3A = 1 \Rightarrow A = -1/3 \\ -3B = 3 \Rightarrow B = -1 \\ -4B - 3C = 4 \Rightarrow C = 0 \\ 2B - 2C - 3D = -5 \Rightarrow D = 1 \\ -7E - 4F = 5 \\ 4E - 7F = 0 \Rightarrow F = \frac{4}{7}E \end{cases}$$

$$y_p = -\frac{1}{3}e^{2x} - x^2 + 1 - \frac{7}{13}\cos(2x) - \frac{4}{13}\sin(2x)$$

$$\Rightarrow 7E + \frac{16}{7}E = -5 \Rightarrow \frac{65}{7}E = -5 \Rightarrow E = -\frac{7}{13} \Rightarrow F = -\frac{4}{13}$$

$\Rightarrow$ General Solution: $\boxed{y = c_1 e^{-x} + c_2 e^{3x} - \frac{1}{3}e^{2x} - x^2 + 1 - \frac{7}{13}\cos(2x) - \frac{4}{13}\sin(2x)}$

Remark: with superposition problems like this, you can also solve each non-homogeneous equation individually and add the results at the end. For example:

$$\left.\begin{array}{l} y'' - 2y' - 3y = e^{2x} \Rightarrow y_{p_1} = -\frac{1}{3}e^{2x} \\ y'' - 2y' - 3y = 3x^2 + 4x - 5 \Rightarrow y_{p_2} = -x^2 + 1 \\ y'' - 2y' - 3y = 5\cos(2x) \Rightarrow y_{p_3} = -\frac{7}{13}\cos(2x) - \frac{4}{13}\sin(2x) \end{array}\right\} \Rightarrow y_p = y_{p_1} + y_{p_2} + y_{p_3}.$$

Case (2): Suppose $g(x) = g_1(x) + \ldots + g_k(x)$, where each function $g_j(x)$ is of the type previously discussed, with corresponding guess y_{p_j} for its particular solution. If any y_{p_j} contains a function in y_c, then <u>multiply this y_{p_j}</u> by the lowest power x^n of x that eliminates the duplication.

Example (2): $\boxed{y'' - y' - 2y = g(x)}$

<u>Complementary Solution</u>: $y_c = c_1 e^{2x} + c_2 e^{-x}$

$m^2 - m - 2 = 0 \Rightarrow (m-2)(m+1) = 0 \Rightarrow m_1 = 2;\ m_2 = -1$

$g(x)$	y_p
$g(x) = e^{3x}$	$y_p = Ae^{3x}$ (no duplication)
$g(x) = e^{2x}$	$y_p = Axe^{2x}$ Original guess: Ae^{2x} duplicates part of y_c Multiplication by x eliminates duplication
$g(x) = 3e^{2x} + \cos(7x)$	$y_p = Axe^{2x} + B\cos(7x) + C\sin(7x)$ <u>Superposition</u>: $g_1(x) = 3e^{2x}$; as above, multiply by x $g_2(x) = \cos(7x)$; original guess $y_{p_2} = B\cos(7x) + C\sin(7x)$ does not duplicate any part of y_c. <u>Careful</u>: $y_p = x\left[Ae^{2x} + B\cos(7x) + C\sin(7x)\right]$ is <u>wrong</u>!
$g(x) = xe^{2x}$	$y_p = (Ax^2 + Bx)e^{2x}$ Original guess: $(Ax+B)e^{2x} = Axe^{2x} + \underbrace{Be^{2x}}_{\text{duplicates } y_c}$ Multiply original guess by x (removes duplication)
$g(x) = \sin(3x)e^{2x}$	$y_p = A\sin(3x)e^{2x} + B\cos(3x)e^{2x}$ (no duplication)
$g(x) = x^2e^{2x} + e^{\pi x}$	$y_p = (Ax^3 + Bx^2 + Cx)e^{2x} + De^{\pi x}$ (Superposition + Duplication)

Example (3): $\boxed{y'' + 16 = g(x)}$

Complementary Solution: $y_c = C_1 \cos(4x) + C_2 \sin(4x)$

$m^2 + 16 = 0$; $m = \pm 4i$

$g(x)$	y_p
$g(x) = x^2 e^x$	$y_p = (Ax^2 + Bx + C)e^x$ (no duplication)
$g(x) = e^x \sin(4x)$	$y_p = Ae^x \sin(4x) + Be^x \cos(4x)$ (no duplication)
$g(x) = 2\cos(4x)$	$y_p = Ax\cos(4x) + Bx\sin(4x)$ Original guess $(A\cos(4x) + B\sin(4x))$ duplicates part of y_c.
$g(x) = x^3\cos(4x) + e^{2x}\sin(4x)$	$y_p = (Ax^4 + Bx^3 + Cx^2 + Dx)\cos(4x)$ $+ (Ex^4 + Fx^3 + Gx^2 + Hx)\sin(4x)$ $+ [Ie^{2x}\sin(4x) + Je^{2x}\cos(4x)]$ Superposition: $g_1(x) = x^3\cos(4x)$, original guess: $(Ax^3 + Bx^2 + Cx + D)\cos(4x) +$ $+ (Ex^3 + Fx^2 + Gx + H)\sin(4x)$ duplicates part of y_c because of the terms $D\cos(4x)$ and $H\sin(4x)$ => multiply by x, duplication eliminated. $g_2(x) = e^{2x}\sin(4x) \Rightarrow y_{P_2} = Ie^{2x}\sin(4x) + Je^{2x}\cos(4x)$ (no duplication)

Example (4): $\boxed{y'' - 2y' + y = 2e^x + x}$

Complementary Solution: $y_c = C_1 e^x + C_2 x e^x$

Particular Solution: $y_p = Ax^2e^x + (Bx + C)$

$g_2(x) = x$; $y_{P_2} = Bx + C$ ✓

$g_1(x) = 2e^x$; $y_{P_1} = Ae^x \cdot 1 \cdot x^2$

duplicates y_c; $xy_{P_1} = Axe^x$ also does

Reduction of Order

Start with a 2^{nd} order linear ODE

$$y'' + P(x)\,y' + Q(x)\,y = 0 \qquad (*)$$

and suppose y_1 is a known solution.

(Assume we are working on an interval I where $y_1(x) \neq 0$ for all $x \in I$.)

Look for another solution, of the form $\boxed{y = u y_1}$ where u is some unknown function of x

$$y = u y_1 \Rightarrow y' = u' y_1 + u y_1'$$
$$\Rightarrow y'' = u'' y_1 + 2u' y_1' + u y_1''$$
$$\Rightarrow y'' + P(x) y' + Q(x) y = u'' y_1 + 2u' y_1' + u y_1'' + P(x) u' y_1 + P(x) u y_1' + Q(x) u y_1$$
$$= u \underbrace{\left(y_1'' + P(x) y_1' + Q(x) y_1\right)}_{0 \text{ b/c } y_1 \text{ is a solution to } (*)} + u'' y_1 + \left(2y_1' + P(x) y_1\right) u'$$

$\Rightarrow y = u y_1$ is a solution to $(*)$ if and only if

$$\boxed{y_1 u'' + (2y_1' + P(x) y_1) u' = 0}$$

Obtain an equation in u of the form:

$$y_1 u'' + (2y_1' + P(x) y_1)\, u' = 0$$

Make the substitution $\boxed{w = u'}$:

$$y_1 w' + (2y_1' + P(x) y_1) w = 0$$

(linear & separable)

Solve and find w

$$w = u' \Rightarrow y_1 w' + (2y_1' + P(x) y_1) w = 0$$
$$\Rightarrow \frac{dw}{dx} + \left(\frac{2y_1'}{y_1} + P(x)\right) w = 0$$
$$\Rightarrow \frac{1}{w}\, dw = -\left(\frac{2y_1'}{y_1} + P(x)\right) dx$$
$$\Rightarrow \ln|w| = -2\ln|y_1| - \int P(x)\,dx + C$$
$$\Rightarrow \ln|w y_1^2| = -\int P(x)\,dx + C$$
$$\Rightarrow w y_1^2 = c\, e^{-\int P(x)dx}$$
$$\Rightarrow \boxed{w = u' = c\, \frac{1}{y_1^2} e^{-\int P(x)dx}}$$

From $w = u'$, find u

$$\Rightarrow \boxed{u = C_2 \int \frac{1}{y_1^2} e^{-\int P(x)dx}\, dx + C_1}$$

Finally, find $y = u y_1$:

$$\Rightarrow y = C_1 y_1 + C_2 \underbrace{y_1 \int \frac{1}{y_1^2} e^{-\int P(x)dx}\, dx}_{y_2 \text{ (new solution)}}$$

You will obtain

$$\boxed{y = C_1 y_1 + C_2 y_2}$$

(the general solution)

(y_2 is another solution, and $\{y_1, y_2\}$ are linearly independent)

Example : Given that $\boxed{y=e^x}$ is a solution to $\boxed{y''-y=0}$, find the general solution.

- Look for a solution $y=ue^x$ for some unknown function of u.

$$y'=u'e^x+ue^x$$

$$y''=u''e^x+2u'e^x+ue^x$$

$$\Rightarrow y''-y=u''e^x+2u'e^x=(u''+2u')e^x$$

$\Rightarrow y=ue^x$ is a solution if and only if $\boxed{u''+2u'=0}$

(because $e^x\neq 0$ for all real x).

- Solve $\boxed{u''+2u'=0}$

Substitution: $\boxed{w=u'}$

$\Rightarrow \boxed{w'+2w=0}$ (Separable)

$$\frac{dw}{dx}=-2w \Rightarrow \frac{1}{w}dw=-2dx$$

$$\Rightarrow \ln|w|=-2x+C \Rightarrow \boxed{w=ce^{-2x}}$$

$$\downarrow w=u'$$

$$u'=ce^{-2x} \Rightarrow u=\int ce^{-2x}dx$$

$$\boxed{u=-\tfrac{1}{2}ce^{-2x}+c_1}$$

- Find $y=ue^x$: $y=ue^x=-\tfrac{1}{2}ce^{-x}+c_1e^x$

$$\Rightarrow \boxed{y=c_1e^x+c_2e^{-x}}$$ General solution

4.7 Variation of Parameters

$$\boxed{y'' + P(x)y' + Q(x)y = g(x)} \to \text{Standard Form}$$

1. Find the complementary solution $y_c = C_1 \underbrace{y_1}_{} + C_2 \underbrace{y_2}_{}$ linearly independent (fundamental set)
2. Compute the determinants:

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \quad \text{(the Wronskian, never 0)}$$

$$W_1 = \begin{vmatrix} 0 & y_2 \\ g(x) & y_2' \end{vmatrix}$$

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & g(x) \end{vmatrix}$$

These come from Cramer's Rule

3. Find the functions $u_1(x), u_2(x)$ given by: $\boxed{u_1' = \frac{W_1}{W} \; ; \; u_2' = \frac{W_2}{W}}$
4. Particular Solution: $\boxed{y_p = u_1 y_1 + u_2 y_2}$
5. General Solution: $y = c_1 y_1 + c_2 y_2 + u_1 y_1 + u_2 y_2$

General Higher Order: $\boxed{y^{(n)} + P_{n-1}(x) y^{(n-1)} + \ldots + P_1(x) y' + P_0(x) y = g(x)}$

1. Find complementary solution: $y_c = C_1 y_1 + C_2 y_2 + \ldots + C_n y_n$
2. Compute the Wronskian of the fundamental set:

$$W = \begin{vmatrix} y_1 & y_2 & \ldots & y_n \\ y_1' & y_2' & \ldots & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \ldots & y_n^{(n-1)} \end{vmatrix}$$

And for every $k \in \{1, 2, \ldots, n\}$, compute the determinant W_k obtained by replacing the k^{th} column of W by: $\begin{bmatrix} 0 \\ \vdots \\ 0 \\ g(x) \end{bmatrix}$

3. Find the functions $u_1(x), \ldots, u_n(x)$ given by: $\boxed{u_k' = \frac{W_k}{W}}$ for every $k \in \{1, \ldots, n\}$
4. Particular Solution: $\boxed{y_p = u_1 y_1 + u_2 y_2 + \ldots + u_n y_n}$
5. General Solution: $y = y_c + y_p$.

Example : $y''' - 2y'' - y' + 2y = e^{3x}$

(1) Complementary Solution :

$$m^3 - 2m^2 - m + 2 = 0$$

$$(m^2-1)(m-2) = 0 \Rightarrow m \in \{-1, 1, 2\}$$

$$y_c = c_1 e^{-x} + c_2 e^{x} + c_3 e^{2x}$$

(2) Determinants :

$$W = \begin{vmatrix} e^{-x} & e^{x} & e^{2x} \\ -e^{-x} & e^{x} & 2e^{2x} \\ e^{-x} & e^{x} & 4e^{2x} \end{vmatrix} \underset{\substack{R_1+R_2 \\ R_3-R_1}}{=} \begin{vmatrix} e^{-x} & e^{x} & e^{2x} \\ 0 & 2e^{x} & 3e^{2x} \\ 0 & 0 & 3e^{2x} \end{vmatrix} = e^{-x} \begin{vmatrix} 2e^{x} & 3e^{2x} \\ 0 & 3e^{2x} \end{vmatrix} = 6e^{2x}$$

$$W_1 = \begin{vmatrix} 0 & e^{x} & e^{2x} \\ 0 & e^{x} & 2e^{2x} \\ e^{3x} & e^{x} & 4e^{2x} \end{vmatrix} = e^{3x} \begin{vmatrix} e^{x} & e^{2x} \\ e^{x} & 2e^{2x} \end{vmatrix} = e^{6x}$$

position a_{32}

$(-1)^{3+2} = (-1)$

$$W_2 = \begin{vmatrix} e^{-x} & 0 & e^{2x} \\ -e^{-x} & 0 & 2e^{2x} \\ e^{-x} & e^{3x} & 4e^{2x} \end{vmatrix} = -e^{3x} \begin{vmatrix} e^{-x} & e^{2x} \\ -e^{-x} & 2e^{2x} \end{vmatrix} = -3e^{4x}$$

$$W_3 = \begin{vmatrix} e^{-x} & e^{x} & 0 \\ -e^{-x} & e^{x} & 0 \\ e^{-x} & e^{x} & e^{3x} \end{vmatrix} = e^{3x} \begin{vmatrix} e^{-x} & e^{x} \\ -e^{-x} & e^{x} \end{vmatrix} = 2e^{3x}$$

(3)

$$u_1' = \frac{e^{6x}}{6e^{2x}} = \frac{1}{6}e^{4x} \Rightarrow \boxed{u_1 = \frac{1}{24}e^{4x}}$$

$$u_2' = \frac{-3e^{4x}}{6e^{2x}} = -\frac{1}{2}e^{2x} \Rightarrow \boxed{u_2 = -\frac{1}{4}e^{2x}}$$

$$u_3' = \frac{2e^{3x}}{6e^{2x}} = \frac{1}{3}e^{x} \Rightarrow \boxed{u_3 = \frac{1}{3}e^{x}}$$

(4)

$$y_p = \left(\frac{1}{24}e^{4x}\right)e^{-x} + \left(-\frac{1}{4}e^{2x}\right)e^{x} + \left(\frac{1}{3}e^{x}\right)e^{2x} = \boxed{\frac{1}{8}e^{3x}}$$

$$y_p = \frac{1}{8}e^{3x}$$

(5)

$$\boxed{y = c_1 e^{-x} + c_2 e^{x} + c_3 e^{2x} + \frac{1}{8}e^{3x}}$$

(4.3) Cauchy-Euler Equation

$$\boxed{ax^2y''+bxy'+cy=0}$$

Solve on $(0,\infty)$

- Characteristic Equation:

$$am^2+(b-a)m+c=0$$

- 3 Cases:

  1. 2 distinct real roots m_1, m_2 : $\boxed{y=c_1x^{m_1}+c_2x^{m_2}}$
  2. 1 repeated real root m_1 : $\boxed{y=c_1x^{m_1}+c_2x^{m_1}\ln x}$
  3. 2 complex roots $\alpha\pm i\beta$: $\boxed{y=x^{\alpha}\left(c_1\cos(\beta\ln x)+c_2\sin(\beta\ln x)\right)}$

Why?

Solve $\boxed{ax^2y''+bxy'+cy=0}$ on $(0,\infty)$.

Substitution : $\boxed{x=e^t, \text{ or } \ln x = t}$

$$\Rightarrow \frac{dy}{dx}=\frac{dy}{dt}\overbrace{\frac{dt}{dx}}^{1/x}=\frac{1}{x}\frac{dy}{dt}$$

$$\Rightarrow \frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dt}\right)$$

$$=-\frac{1}{x^2}\frac{dy}{dt}+\frac{d}{dx}\frac{dy}{dt}\cdot\frac{1}{x}$$

$$=-\frac{1}{x^2}\frac{dy}{dt}+\frac{d^2y}{dt^2}\cdot\frac{dt}{dx}\cdot\frac{1}{x}$$

$$=\frac{1}{x^2}\left(\frac{d^2y}{dt^2}-\frac{dy}{dt}\right)$$

$$ax^2\cdot\frac{1}{x^2}\left(\frac{d^2y}{dt^2}-\frac{dy}{dt}\right)+bx\cdot\left(\frac{1}{x}\frac{dy}{dt}\right)+cy=0$$

$$a\frac{d^2y}{dt^2}-a\frac{dy}{dt}+b\cdot\frac{dy}{dt}+cy=0$$

$$\boxed{a\frac{d^2y}{dt^2}+(b-a)\frac{dy}{dt}+cy=0}$$

Constant Coeff. w/ Char. eqn.:

$$\boxed{am^2+(b-a)m+c=0}$$

- 2 distinct real roots m_1, m_2 $\Rightarrow y=c_1e^{m_1t}+c_2e^{m_2t}\underset{(t=\ln x)}{=}c_1x^{m_1}+c_2x^{m_2}$
- 1 repeated real root m_1 $\Rightarrow y=c_1e^{m_1t}+c_2te^{m_1t}=c_1x^{m_1}+c_2x^{m_1}\ln x.$
- 2 complex roots $\alpha\pm i\beta$ $\Rightarrow y=e^{\alpha t}\left(c_1\cos(\beta t)+c_2\sin(\beta t)\right)$

$$=x^{\alpha}\left(c_1\cos(\beta\ln x)+c_2\sin(\beta\ln x)\right).$$