Classifications of DEs:

① By Type

Ordinary Differential Equations (ODE) (a single independent variable)

Ex: $\frac{du}{dx} - \frac{d^2v}{dx^2} = x$; $\frac{d^3y}{dt^3} - 2\frac{dy}{dt} + 3t = 0$

Partial Differential Equations (PDE) (2 or more independent variables)

Ex: $\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}$; $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial u}{\partial y}$

② By Order:

The order of a DE is the order of the highest-order derivative.

Ex: $\frac{d^2y}{dt^2} + 5\left(\frac{dy}{dt}\right)^3 - 4y = e^x$ Second Order ODE

$4\frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial x \partial y} = 0$ Fourth Order PDE

③ By Linearity:

An n^{th} order ODE is linear if it can be written in the form:

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \dots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

Characteristics:

- The dependent variable and all of its derivatives are raised to the power 1.
- Each coefficient $(a_n, a_{n-1}, \dots, a_0, g)$ depends only on the independent variable.

Ex: $x\frac{dy}{dx} + y = 0$ Linear, 1^{st} order

Ex: $x^3\frac{d^2y}{dx^2} + \sin(x)\frac{dy}{dx} = e^x$ Linear 2^{nd} order

Ex: $e^{\cos x}y^{(6)} + 7^x y' + xy = 0$ Linear 7^{th} order

Ex: $yy'' - 2y' = x$ Non-linear, 2^{nd} order.

Ex: $y^{(3)} + (y')^2 + y = 0$ Non-linear, 3^{rd} order.

Ex: $y' = \frac{y}{y-x}$ Non-linear (in y), 1^{st} order [Reciprocal: $x' = \frac{y-x}{y} = 1 - \frac{1}{y}x$ is linear in x]

Ex: $y' = \frac{xy}{y-x}$ Non-linear (in y), 1^{st} order [Reciprocal: $x' = \frac{y-x}{xy}$, $x' = \frac{1}{x} - \frac{1}{y}$ non-linear in x]

Methods for Solving 1^{st} Order ODEs

- **Separable Equations**: $\boxed{\dfrac{dy}{dx} = \dfrac{g(x)}{h(y)}}$

Method: $\int h(y)\,dy = \int g(x)\,dx$

Integrate & solve for y (or get an implicit solution)

Ex: $(1+x)dy - y\,dx = 0$

$$(1+x)\,dy = y\,dx$$
$$\frac{1}{y}\,dy = \frac{1}{1+x}\,dx$$
$$\ln|y| = \ln|1+x| + C$$
$$|y| = c\,|1+x| \Rightarrow y = \pm c(1+x)$$
$$\boxed{y = c(1+x)}$$

- **Autonomous Equations**: $\boxed{\dfrac{dy}{dx} = f(y)}$
  - * Separable: If $f(y) \neq 0$; $\frac{1}{f(y)}dy = dx$
  - * Critical Points: c s.t. $f(c) = 0$
  - * Equilibrium Solutions: $y = c$ where c = critical pt.
  - * Phase portraits

- **Linear Equations**: $\boxed{\dfrac{dy}{dx} + p(x)y = g(x)}$ (Standard Form)

Method: Find Integrating Factor: $\mu(x) = e^{\int p(x)dx}$ (from Standard Form)

Multiply eqn. by $\mu(x)$

$\Rightarrow$ Eqn. becomes $\frac{d}{dx}(y\mu(x)) = g(x)\mu(x)$

Integrate: $y\mu(x) = \int g(x)\mu(x)\,dx$

Solve for y.

Ex: $\dfrac{dy}{dx} + 3y = x$

$$\mu(x) = e^{\int 3dx} = e^{3x}$$
$$e^{3x}y' + 3e^{3x}y = xe^{3x}$$
$$\frac{d}{dx}(e^{3x}y) = xe^{3x}$$
$$e^{3x}y = \int xe^{3x} = \frac{1}{3}\int x(e^{3x})'dx$$
$$= \frac{1}{3}\left(xe^{3x} - \int e^{3x}dx\right)$$
$$= \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + C$$
$$\boxed{y = \frac{1}{3}x - \frac{1}{9} + Ce^{-3x}}$$

- **Exact Equations:** $M(x,y)dx + N(x,y)dy = 0$; $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Method: Find potential: $f(x,y)$ s.t. $\frac{\partial f}{\partial x} = M$; $\frac{\partial f}{\partial y} = N$

$\Rightarrow$ Eqn. becomes $\underbrace{\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}}$ $= 0$

Chain Rule: $\frac{df}{dx} = 0 \Rightarrow \boxed{f(x,y) = c}$ Solution

Ex): $2xy\,dx + (x^2-1)dy = 0$

$\frac{\partial M}{\partial y} = 2x$; $\frac{\partial N}{\partial x} = 2x \Rightarrow$ exact

Potential: $\frac{\partial f}{\partial x} = 2xy \Rightarrow f(x,y) = x^2y + g(y)$

$\Rightarrow \left.\begin{aligned}\frac{\partial f}{\partial y} &= x^2 + g'(y)\\ &= x^2 - 1\end{aligned}\right\} \Rightarrow g'(y) = -1 \Rightarrow g(y) = -y$

$\Rightarrow f(x,y) = x^2y - y \Rightarrow (x^2-1)y = c \Rightarrow \boxed{y = \frac{c}{x^2-1}}$

- **Homogenous Eqns.:** $M(x,y)dx + N(x,y)dy = 0$; M, N homogeneous of same degree:

$M(\alpha x, \alpha y) = \alpha^k M(x,y)$

$N(\alpha x, \alpha y) = \alpha^k N(x,y)$

Method: Either one of the substitutions $\underbrace{y = ux}_{dy = u\,dx + x\,du}$ or $\underbrace{x = uy}_{dx = u\,dy + y\,du}$ turns the eqn. separable.

OR: write eqn. as $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$ & make the substitution $\underbrace{u = \frac{y}{x}}_{\frac{dy}{dx} = u + x\frac{du}{dx}}$ $\Rightarrow$ separable

OR: write eqn. as $\frac{dx}{dy} = F\left(\frac{x}{y}\right)$ & make the substitution $\underbrace{u = \frac{x}{y}}_{\frac{dx}{dy} = u + y\frac{du}{dy}}$ $\Rightarrow$ separable

Ex: $(x^2+y^2)\,dx+(x^2-xy)\,dy=0$ (homogeneous of degree 2)

$\boxed{y=ux}$ $\Rightarrow$ $(x^2+u^2x^2)dx+(x^2-x^2u)(u\,dx+x\,du)=0$

$dy=u\,dx+x\,du$

$$(1+u^2)dx+(1-u)(u\,dx+x\,du)=0$$
$$(1+u)\,dx=(u-1)x\,du$$
$$\frac{1}{x}dx=\frac{u-1}{u+1}du \Rightarrow \ln|x|=\int\frac{u+1-2}{u+1}du=\int 1-\frac{2}{u+1}\,du$$
$$=u-2\ln|u+1|+C$$

OR: Divide by x^2 and write eqn. as:

$$(1+(y/x)^2)+(1-y/x)\frac{dy}{dx}=0$$

$\boxed{u=y/x}$ $\Rightarrow$ $(1+u^2)+(1-u)\left(u+x\frac{du}{dx}\right)=0$

$\frac{dy}{dx}=u+x\frac{du}{dx}$

$$1+u=(u-1)x\frac{du}{dx}$$
$$(1+u)dx=(u-1)x\,du$$
$$\vdots$$

$$\Rightarrow \ln|x|+2\ln|u+1|-u=C$$
$$\ln|x(u+1)^2|-u=C$$
$$\ln\left|x\left(\frac{y}{x}+1\right)^2\right|-\frac{y}{x}=\ln|c|$$
$$\ln\left|\frac{(x+y)^2}{cx}\right|=+\frac{y}{x} \Rightarrow \frac{(x+y)^2}{cx}=e^{y/x}$$
$$\boxed{(x+y)^2=cxe^{y/x}}$$

- **Bernoulli Equations**: $\boxed{\frac{dy}{dx}+P(x)y=Q(x)y^\alpha}$ $\alpha\in\mathbb{R}$

Method: If $\alpha=0, \alpha=1 \Rightarrow$ linear

If $\boxed{\alpha\neq 1}$: substitution $\boxed{u=y^{1-\alpha}}$ $\Rightarrow \frac{du}{dx}=(1-\alpha)y^{-\alpha}\frac{dy}{dx} \Rightarrow y^{-\alpha}\frac{dy}{dx}=\frac{1}{1-\alpha}\frac{du}{dx}$

Divide eqn. by y^α: $\underbrace{y^{-\alpha}\frac{dy}{dx}}+P(x)\underbrace{y^{1-\alpha}}=Q(x)$

$$\frac{1}{1-\alpha}\frac{du}{dx}+P(x)u=Q(x) \Rightarrow \boxed{\frac{du}{dx}+(1-\alpha)P(x)u=(1-\alpha)Q(x)}$$

(linear)

Solve for u,
use u to find y.

Ex: $y'=\underset{P(x)}{5}y+\underset{Q(x)}{e^{-2x}}y^{-2}$; $y'-5y=e^{-2x}y^{-2}$ Bernoulli w/ $\alpha=-2$

$1-\alpha=3 \Rightarrow \boxed{u=y^3}$

$$\Rightarrow u'+3(-5)u=3\cdot e^{-2x}$$
$$\Rightarrow u'-15u=3e^{-2x}$$
$$\mu(x)=e^{-15x} \Rightarrow \frac{d}{dx}\left(ue^{-15x}\right)=3e^{-17x}$$
$$\Rightarrow ue^{-15x}=-\frac{3}{17}e^{-17x}+c \Rightarrow u=\boxed{y^3=-\frac{3}{17}e^{-2x}+ce^{15x}}$$

FIRST ORDER ODEs

$$\frac{dy}{dx} = f(x,y)$$

Autonomous: $\frac{dy}{dx} = f(y)$

Separable: $\frac{dy}{dx} = \frac{g(x)}{h(y)}$

Method: Separate, Integrate, Solve

$$\int h(y)\,dy = \int g(x)\,dx$$

Exact: $M(x,y)dx + N(x,y)dy = 0$; $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Method: Find potential: $f(x,y)$ s.t. $\frac{\partial f}{\partial x} = M$; $\frac{\partial f}{\partial y} = N$

$\Rightarrow$ Solution: $f(x,y) = c$

Linear: $\frac{dy}{dx} + p(x)y = g(x)$

Method: Find integrating factor: $\mu(x) = e^{\int p(x)dx}$

Multiply eqn. by $\mu(x)$ & it becomes:

$$\frac{d}{dx}(y\mu(x)) = \mu(x)g(x)$$

Integrate: $y\mu(x) = \int \mu(x)g(x)dx$

Solve for y.

Homogeneous: $M(x,y)dx + N(x,y)dy = 0$

M, N homogeneous of same degree:

$M(\alpha x, \alpha y) = \alpha^k M(x,y)$

$N(\alpha x, \alpha y) = \alpha^k N(x,y)$

Method: Substitution: Either one of the substitutions:

$\boxed{y = ux}$ $\rightsquigarrow$ $\boxed{dy = x\,du + u\,dx}$

$\boxed{u = \frac{y}{x}}$

turns it into a separable ODE

Bernoulli: $\frac{dy}{dx} + p(x)y = Q(x)y^{\alpha}$; $\alpha \in \mathbb{R}$

Method: $\alpha = 0$ or $\alpha = 1 \Rightarrow$ linear

$\boxed{\alpha \neq 1}$: Substitution $\boxed{u = y^{1-\alpha}}$

$\Rightarrow$ Eqn. becomes: $\boxed{\frac{du}{dx} + (1-\alpha)P(x)u = (1-\alpha)Q(x)}$

(linear)