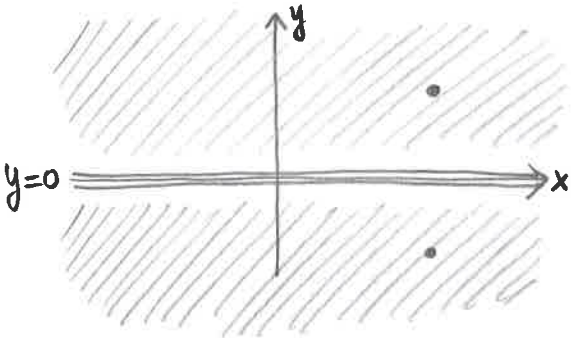
1 (a). $y' = y^{2/3}$

$y' = f(x,y)$; $f(x,y) = y^{2/3}$ → continuous on $\mathbb{R}^2$

$$\frac{\partial f}{\partial y} = \frac{2}{3} y^{-1/3} = \frac{2}{3\sqrt[3]{y}}$$ → continuous everywhere except $y=0$

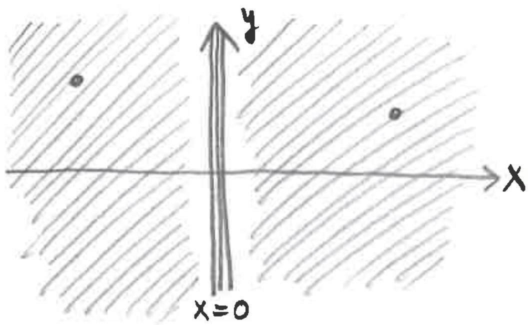


Possible regions :

- The upper half-plane ($y>0$)
- The lower half-plane ($y<0$)

(b). $xy' = y$; $y' = f(x,y)$; $f(x,y) = \frac{y}{x}$ → continuous everywhere except $x=0$

$$\frac{\partial f}{\partial y} = \frac{1}{x}$$ → same

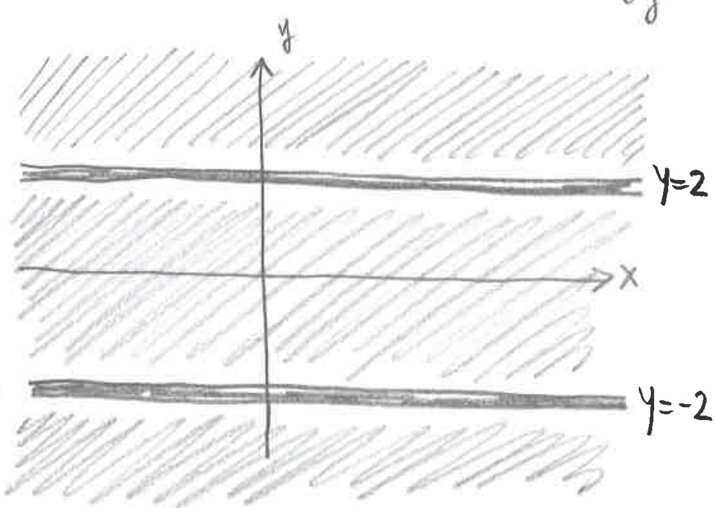


Possible regions :

- The left half-plane ($x<0$)
- The right half-plane ($x>0$)

(c). $(4-y^2)y' = x^2$; $y' = f(x,y)$; $f(x,y) = \frac{x^2}{4-y^2}$ → continuous everywhere except $y = \pm 2$

$$\frac{\partial f}{\partial y} = + \frac{2x^2 y}{(4-y^2)^2}$$ → same

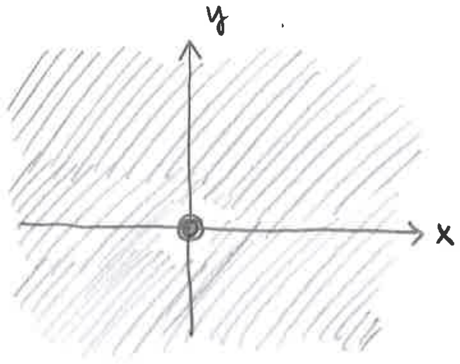


Possible regions :

- $y>2$
- $-2<y<2$
- $y<-2$

(d). $(x^2+y^2)y' = y^2$; $y' = f(x,y)$; $f(x,y) = \dfrac{y^2}{x^2+y^2}$ → continuous everywhere except where $x^2+y^2=0$, i.e. $(0,0)$.

$$\frac{\partial f}{\partial y} = \frac{2yx^2}{(x^2+y^2)^2} \rightsquigarrow \text{same}$$



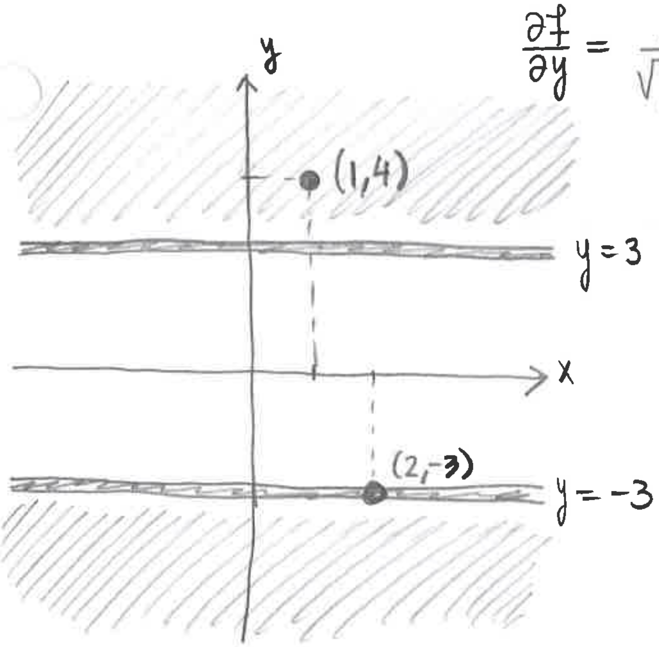
Possible regions:

- Any region not containing $(0,0)$.

2. $y' = \sqrt{y^2-9}$

$y' = f(x,y)$; $f(x,y) = \sqrt{y^2-9}$ ⇝ continuous everywhere where defined ($y \leq -3$ or $y \geq 3$)

$$\frac{\partial f}{\partial y} = \frac{y}{\sqrt{y^2-9}} \rightsquigarrow \text{same, except } (y<-3 \text{ or } y>3)$$



(a). $(1,4)$: Yes, because the point lies in a region where both f and f_y are continuous.

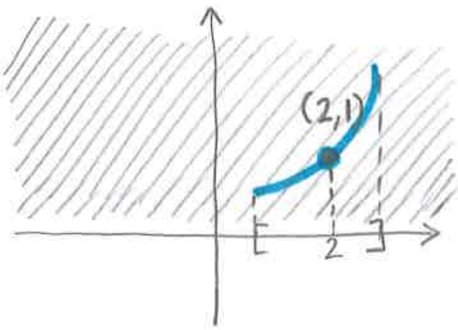
(b). $(2,-3)$: No, because any rectangle containing $(2,-3)$ will contain part of the line $y=-3$, where $\frac{\partial f}{\partial y}$ does not even exist.

Note: This does not imply necessarily that the IVP: $y' = \sqrt{y^2-9}$; $y(2) = -3$ does not have a unique solution, just that we don't know what happens (based on Theorem 1 alone).

③ $y' = x\sqrt{y}$; $y(2) = 1$.

$y' = f(x,y)$; $f(x,y) = x\sqrt{y}$ $\rightsquigarrow$ continuous on $\mathbb{R} \times [0,\infty)$

$$\frac{\partial f}{\partial y} = \frac{x}{2\sqrt{y}} \rightsquigarrow \text{continuous on } \mathbb{R} \times (0,\infty)$$



The point $(2,1)$ lies in the interior of a region where both f and f_y are continuous, so Theorem 1 guarantees that there is some interval $(2-h, 2+h)$ about $x=2$, where the IVP has a unique solution.

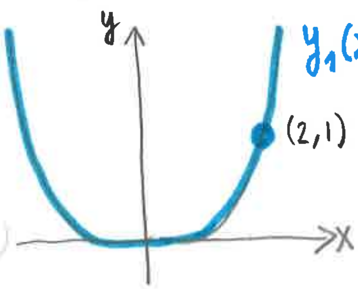
$$\left.\begin{aligned} y_1(x) &= \frac{1}{16}x^4 \Rightarrow x\sqrt{y_1(x)} = x\sqrt{\frac{1}{16}x^4} = x \cdot \frac{1}{4}x^2 = \frac{1}{4}x^3 \\ y_1'(x) &= \frac{1}{4}x^3 \end{aligned}\right\} \Rightarrow \boxed{\begin{gathered} y_1'(x) = x\sqrt{y_1'(x)} \\ \text{for all } x \in \mathbb{R} \end{gathered}}$$

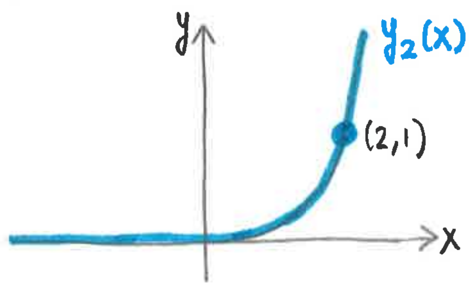
✓

$$\left.\begin{aligned} y_2(x) &= \begin{cases} 0, & \text{if } x<0 \\ \frac{1}{16}x^4, & \text{if } x \geqslant 0 \end{cases} \Rightarrow x\sqrt{y_2(x)} = \begin{cases} 0, & \text{if } x<0 \\ \frac{1}{4}x^3, & \text{if } x \geqslant 0 \end{cases} \\ y_2'(x) &= \begin{cases} 0, & \text{if } x<0 \\ \frac{1}{4}x^3, & \text{if } x \geqslant 0 \end{cases} \end{aligned}\right\} \Rightarrow \boxed{\begin{gathered} y_2'(x) = x\sqrt{y_2'(x)} \\ \text{for all } x \in \mathbb{R} \end{gathered}}$$

✓

Why is there no contradiction then, between these examples, and what we deduced from Theorem 1?





Because all Thm. 1 guarantees is a unique solution <u>in some interval centered at $x=2$</u>

– the two functions coincide on $(0,\infty)$, so there is no contradiction.